Representation and Characterization of Quasistationary Distributions for Markov Chains

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Abstract

This work provides a characterization of Quasistationary Distributions (QSDs) for Markov chains with a unique absorbing state and an irreducible set of non-absorbing states. As is well known, every QSD has an associated absorption parameter describing the exponential tail of the absorption time under the law of the process with the QSD as the initial distribution. The analysis associated with the existence and representation of QSDs corresponding to a given parameter is according to whether the moment generating function of the absorption time starting from any non-absorbing state evaluated at the parameter is finite or infinite, the *finite* or *infinite moment generating function regimes*, respectively. For parameters in the finite regime, it is shown that when they exist, all QSDs are in the convex cone of a Martin entry boundary associated with the parameter. The infinite regime corresponds to at most one parameter value and at most one QSD. In this regime, when a QSD exists, it is unique and can be represented by a renewal-type formula. Several applications to the findings are presented, including revisiting previously established results using the developments in this work.

1 Introduction

Let $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ the set of nonnegative integers, and $\mathbb{N} = \{1, 2, 3, \dots\}$ the set of natural numbers. We also write $\mathbb{R}_+$ for the set of nonnegative real numbers.

Consider a Markov chain $\mathbf{X} = (X_n : n \in \mathbb{Z}_+)$ on a state space which is a disjoint union of the set S and the singleton $\{\Delta\}$, and where S is either finite or countably infinite. Let p denote the transition function for $\mathbf{X}$. As usual, we write P_μ and E_μ for the probability and expectation associated with $\mathbf{X}$ under

the initial distribution μ, with P_x and E_x serving as shorthand for P_{δ_x} and E_{δ_x}, respectively.

For $x \in S \cup \{\Delta\}$, let

$$\tau_x = \inf\{n \in \mathbb{N} : X_n = x\}, \tag{1.1}$$

and

$$^0\tau_x = \inf\{n \in \mathbb{Z}_+ : X_n = x\}. \tag{1.2}$$

We work under the following additional hypotheses:

HD-1. *$\tau_\Delta < \infty$ P_x-a.s. for some $x \in S$.*

HD-2. *The restriction of p to S is irreducible.*

As a result, Δ is a unique absorbing state. We therefore refer to τ_Δ as the absorption time.

Definition 1.1 (QSD)**.** *Let* **HD-1,2** *hold. A probability measure ν on S is a Quasistationary Distribution (QSD) if*

$$P_\nu(X_n \in \cdot \mid \tau_\Delta > n) = \nu(\cdot) \quad \textit{for all} \quad n \in \mathbb{Z}_+. \tag{1.3}$$

As is well-known, [28, Theorem 2] ν is a QSD if and only if it is a "Quasi limiting distribution" in the sense that there exists some probability measure μ on S so that

$$\lim_{n\to\infty} P_\mu(X_n \in \cdot \mid \tau_\Delta > n) = \nu. \tag{1.4}$$

The idea of a limiting conditional distribution traces back to as early as 1931 by Wright [30] in the discussion of gene frequencies in finite populations. Later, Bartlett introduced the notion of "quasi stationarity" [3] and coined the term "quasi-stationary distribution" in the context of a birth and death process [4]. Yaglom [31] was the first who showed explicitly that the (1.4) holds for a non-degenerate subcritical branching process starting from any deterministic initial population, with a limit independent of the initial population. To this day, results of this type are referred to as Yaglom limits.

An important property of a QSD is the following [22, Theorem 2.2]. Suppose ν is a QSD. Then there exists some $\lambda > 0$

$$P_\nu(\tau_\Delta > n) = e^{-\lambda n}. \tag{1.5}$$

That is under P_ν, τ_Δ is geometric with parameter $1 - e^{-\lambda}$. We refer to λ as the absorption parameter associated with ν.

The definition of a QSD immediately leads to the following characterization of QSD, see e.g., [22].

Proposition 1.1. *Let* **HD-1,2** *hold. A probability measure ν on S is a QSD if and only if there exists some $\lambda > 0$ such that*

$$\sum_{i\in S} \nu(i)p(i,j) = e^{-\lambda}\nu(j), \;\; j \in S. \tag{1.6}$$

In this case, ν is a QSD with absorption parameter λ.

Two comments are in place:

1. Since S is irreducible, if ν is a QSD then $\nu(i) > 0$ for all $i \in S$.
2. The analogous statement to Proposition 1.1 in the continuous-time setting is generally false, see [27, Section 3.3].

In light of (1.5), the following is a necessary condition for the existence of a QSD:

HD-3. *There exists $\beta > 0$ such that $E_x[\exp(\beta\tau_\Delta)] < \infty$ for some (equivalently all) $x \in S$.*

We note that the statement in parenthesis holds due to the assumed irreducibility of p. Clearly, **HD-3** is not a sufficient condition, as the following very simple example shows:

Example 1.1. *Let $\mathfrak{p}$ be any irreducible transition function on S. Let p be defined as follows. Fix $\lambda_{cr} > 0$, and define a transition function p on $S \cup \{\Delta\}$ by letting $p(x, y) = e^{-\lambda_{cr}}\mathfrak{p}(x, y)$ when $x, y \in S$ and $p(x, \Delta) = 1 - e^{-\lambda_{cr}}$. In terms of sample paths, $\mathfrak{p}$ and p are related as follows. Let $\mathbf{Y}$ be a MC corresponding to $\mathfrak{p}$, and let τ_Δ be a geometric random variable with parameter $1-e^{-\lambda_{cr}}$, independent of $\mathbf{Y}$. Now for $t \in \mathbb{Z}_+$ set*

$$X_n = Y_n \mathbf{1}_{\{\tau_\Delta > n\}}.$$

Then $\mathbf{X} = (X_n : n \in \mathbb{Z}_+)$ is a MC with transition function p, and (1.5) holds with $\lambda = \lambda_{cr}$ and any distribution ν on S. However, for every probability measure ν on S, $P_\nu(X_n \in \cdot \mid \tau_\Delta > n) = P_\nu(Y_n \in \cdot)$ and so ν is a QSD for p if and only if it is a stationary distribution for $\mathfrak{p}$, and this holds if and only if $\mathfrak{p}$ is positive recurrent.

Definition 1.2. *Let* **HD-1,2,3** *hold.*

1. *The critical absorption parameter λ_{cr} is given by*

$$\lambda_{cr} = \sup\{\lambda > 0 : E_x[e^{\lambda\tau_\Delta}] < \infty \text{ for some } x \in S\}, \tag{1.7}$$

2. *A parameter $\lambda \in (0, \lambda_{cr}]$ is in the finite MGF regime if $E_x[\exp(\lambda\tau_\Delta)] < \infty$ for some $x \in S$.*
3. *The critical absorption parameter λ_{cr} is in the infinite MGF regime if $E_x[\exp(\lambda_{cr}\tau_\Delta)] = \infty$ for some $x \in S$.*

The constant $e^{\lambda_{cr}}$ also appears in the literature [8] as the convergence parameter for p, and the constant $e^{-\lambda_{cr}}$ is also known convergence norm, a generalization of the Perron-Frobenius root. The infinite MGF regime is referred to as the R-recurrent case with $R = e^{\lambda_{cr}}$. We decided not to use these terms as our analysis covers all absorption parameters, not just λ_{cr}, and also because R-recurrence is not a criterion for the existence or non-existence of QSDs but rather an indicator for which approach to apply, something which is simpler to describe through the two regimes presented.

As an immediate corollary, we have the following:

Corollary 1.1. *Let* **HD-1,2,3** *hold. Then*

1. $\lambda_{cr} \in (0, \infty)$
2. *If ν is a QSD with absorption parameter λ, then $\lambda \leq \lambda_{cr}$.*

Proof. Clearly **HD-3** implies $\lambda_{cr} > 0$, and the irreducibility of S implies that for every $x \in S$, $p^n(x,x) > 0$, hence - by induction - $P_x(\tau_\Delta > kn) \geq (p^n(x,x))^k$, which in turn implies $e^{\lambda_{cr}} \leq (p^n(x,x))^{-1/n}$ and so $\lambda_{cr} < \infty$.

For the second assertion, if ν is a QSD, with absorption parameter λ, (1.5) implies that for any $\epsilon > 0$, $E_x[(e^\lambda/(1+\epsilon))^{\tau_\Delta}] < \infty$, and the result follows. □

If ν is a QSD with absorption parameter λ then

$$E_\nu[\tau_\Delta] \overset{(1.5)}{=} \frac{1}{1-e^{-\lambda}} \overset{\text{Cor. 1.1}}{\geq} \frac{1}{1-e^{-\lambda_{cr}}}.$$

For this reason, a QSD with absorption parameter λ_{cr} is called a minimal QSD: it minimizes the expected absorption time among all initial distributions that are QSDs.

We close this section with the following observation.

Proposition 1.2. *Let $\lambda \in (0, \lambda_{cr}]$. Then for every $x \in S$, $E_x[\exp(\lambda\tau_x), \tau_x < \tau_\Delta] \leq 1$. Moreover,*

1. *If $E_x[\exp(\lambda\tau_\Delta)] < \infty$ then the inequality is strict;*
2. *If $E_x[\exp(\lambda_{cr}\tau_\Delta)] = \infty$ and $E_x[\exp(\lambda_{cr}\tau_\Delta), \tau_\Delta < \tau_x] < \infty$ for some $x \in S$, then $E_x[\exp(\lambda_{cr}\tau_x), \tau_x < \tau_\Delta] = 1$ for all $x \in S$.*

Proof. Pick $\lambda < \lambda_{cr}$. Then $E_x[\exp(\lambda\tau_\Delta)] < \infty$. Partitioning according to $\tau_x < \tau_\Delta$ or $\tau_x > \tau_\Delta$ and using the strong Markov property, we have

$$E_x[\exp(\lambda\tau_\Delta)] = E_x[\exp(\lambda\tau_x), \tau_x < \tau_\Delta]E_x[\exp(\lambda\tau_\Delta)] + E_x[\exp(\lambda\tau_\Delta), \tau_\Delta < \tau_x].$$

As the left-hand side is finite and the second term on the right-hand side is strictly larger than zero, the first statement holds, with all terms on the right-hand side finite. Moreover, we can write

$$E_x[\exp(\lambda\tau_\Delta)] = \frac{E_x[\exp(\lambda\tau_\Delta), \tau_\Delta < \tau_x]}{1 - E_x[\exp(\lambda\tau_x), \tau_x < \tau_\Delta]}. \tag{1.8}$$

Both assertions now follow from the monotone convergence theorem by letting $\lambda \uparrow \lambda_{cr}$. □

2 Results: Discrete-Time

2.1 Infinite MGF Regime

Throughout this section, we assume that the hypotheses **HD-1,2,3** hold.

The main result of this section is a necessary and sufficient condition for the existence and uniqueness of a minimal QSD in the infinite MGF regime.

Theorem 2.1. *Suppose λ_{cr} is in the infinite MGF regime. Then*

1. *There exists a minimal QSD if and only if for some $x \in S$*

$$E_x[\exp(\lambda_{cr}\tau_\Delta \wedge \tau_x)] < \infty \tag{2.1}$$

In this case, there exists a unique minimal QSD, given by the formula

$$\nu_{cr}(x) = \frac{e^{\lambda_{cr}} - 1}{E_x[\exp(\lambda_{cr}\tau_\Delta), \tau_\Delta < \tau_x]}, \ x \in S. \tag{2.2}$$

2. *If, in addition to* (2.1),

$$E_x[\exp(\lambda_{cr}\tau_x)\tau_x, \tau_x < \tau_\Delta] < \infty \text{ for some } x \in S, \tag{2.3}$$

and p is aperiodic, then (1.4) *holds for any finitely supported μ, with $\nu = \nu_{cr}$.*

As mentioned below Definition 1.2 λ_{cr} is in the infinite MGF regime if and only if p is R-recurrent with $R = e^{\lambda_{cr}}$ [8][9]. In this regime, p is called R-recurrent if (2.3) holds, and under this additional assumption, Theorem D of [8] provides an infinite dimensional version of Perron-Frobenius: existence of unique (up to scalar multiples) of left and right eigenvectors for p with eigenvalue $1/R$, as well as convergence. However, R-positive recurrence does not imply the existence of a QSD as, in general, the left eigenvector cannot be normalized to a probability measure. The main result of [9] gives conditions that imply R-positive recurrence with a left eigenvector, which is a minimal QSD and convergence results. Nevertheless, in the infinite MGF regime, R-positive recurrence is not necessary for the existence of a minimal QSD, see Proposition 8.2. Moreover, minimal QSDs also exist in the finite MGF regime, see Proposition 8.1 and Theorem 8.1.

Next, we provide a condition equivalent to (2.1), which may be easier to verify directly in some cases. For $K \subsetneq S$, define the hitting time

$$\tau_K = \inf\{n \in \mathbb{N} : X_n \in K\}. \tag{2.4}$$

Proposition 2.1. *Suppose that $K \subsetneq S$ is nonempty and finite and that for some $x \notin K$,*

$$E_x[\exp(\lambda_{cr}\tau_\Delta \wedge \tau_K)] < \infty. \tag{2.5}$$

Then there exists $z \in K$ so that

$$E_z[\exp(\lambda_{cr}\tau_\Delta \wedge \tau_z)] < \infty.$$

The next result is a weak version in discrete time of the main result in [19]. The main differences are that the authors worked in continuous time, did not assume irreducibility, and gave an explicit geometric bound on the convergence rate without identifying the QSD. We expand this discussion when we present a continuous-time version, Theorem 7.2.

Theorem 2.2. *Suppose that there exists some $\bar{\lambda} > 0$ and a nonempty finite $K \subsetneq S$ such that both following conditions hold:*

$$E_x[\exp(\bar{\lambda}\tau_\Delta)] = \infty \text{ for some } x \in S; \tag{2.6}$$

$$\sup_{x \notin K} E_x[\exp(\bar{\lambda}\tau_\Delta \wedge \tau_K)] < \infty. \tag{2.7}$$

Then

1. *$\lambda_{cr} \in (0, \bar{\lambda}]$, λ_{cr} is in the infinite regime, and (2.1), (2.3) and (2.5)hold.*
2. *There exists a unique QSD. This QSD is minimal and is given by (2.2).*

If, in addition, p is aperiodic and there exists some $x_0 \in S$ and $n_0 \in \mathbb{N}$ such that

$$\inf_{x \in S} \frac{P_x(\tau_{x_0} < \tau_\Delta)}{P_x(\tau_\Delta > n_0)} > 0, \tag{2.8}$$

then (1.4) holds for any initial distribution μ.

Proposition 2.2. *Suppose S is finite. Then (2.6), (2.7) and (2.8) hold with $\bar{\lambda} = \lambda_{cr}$ and $K = S - \{x_0\}$, where $x_0 \in S$ is any element maximizing $S \ni x \to p(x, \Delta)$.*

2.2 Finite MGF Regime

The finite state space case is settled by Proposition 2.2. Therefore, in addition to **HD-1,2,3**, in this section, we will also impose the following:

HD-0. *S is countably infinite.*

2.2.1 Martin Boundary Representation

Definition 2.1. *Let $\lambda > 0$ be in the finite MGF regime. For $x \in S$, define*

1. *The Green's function*

$$\begin{aligned} G^\lambda(x, y) &= E_x[\sum_{0 \le s < \tau_\Delta} \exp(\lambda s)\delta_y(X_s)] \\ &= \frac{E_x[\exp(\lambda\tau_y), \tau_y < \tau_\Delta]}{1 - E_y[\exp(\lambda\tau_y), \tau_y < \tau_\Delta]}. \end{aligned} \tag{2.9}$$

   Then $G^\lambda(x, \cdot)$ is a finite measure on S with total mass

$$G^\lambda(x, \mathbf{1}) = \frac{E_x[\exp(\lambda\tau_\Delta)] - 1}{e^\lambda - 1}.$$

2. *The normalized kernel $K^\lambda(x, \cdot)$, a probability measure on S in the second variable,*

$$K^\lambda(x, y) = \frac{G^\lambda(x, y)}{G^\lambda(x, \mathbf{1})} = \frac{e^\lambda - 1}{E_x[\exp(\lambda\tau_\Delta)] - 1} \times \frac{E_x[\exp(\lambda\tau_y), \tau_y < \tau_\Delta]}{1 - E_y[\exp(\lambda\tau_y), \tau_y < \tau_\Delta]}. \tag{2.10}$$

3. *A sequence* $\mathbf{x} = (x_n : n \in \mathbb{N})$ *of elements in* S *is* λ*-convergent if for every* $y \in S$, $\lim_{n\to\infty} K^\lambda(x_n, y)$ *exists. If* $\mathbf{x}$ *is* λ*-convergent, we denote the limit (probability or sub-probability) measure by* $K^\lambda(\mathbf{x}, \cdot)$.

4. *A sequence* $\mathbf{x}$ *is* λ, ∞*-convergent if it is* λ*-convergent and* $\lim_{n\to\infty} x_n = \infty$.

Note that any eventually constant sequence in S is λ-convergent, but not λ, ∞-convergent and that due to the fact $K^\lambda(\cdot, \cdot) \in (0, 1]$ and a diagonal argument, every unbounded sequence has a λ, ∞-convergent subsequence.

Definition 2.2 (Martin Compactification)**.**

1. *The* λ, ∞*-convergent sequences* $\mathbf{x}$ *and* $\mathbf{x}'$ *are* λ*-equivalent if* $K^\lambda(\mathbf{x}, \cdot) = K^\lambda(\mathbf{x}', \cdot)$*, writing* $[\mathbf{x}]$ *for the equivalence class and* $K^\lambda([\mathbf{x}], \cdot)$ *for* $K^\lambda(\mathbf{x}, \cdot)$.

2. *The Martin Boundary* $\partial^\lambda M$ *is the set of equivalence classes of* λ, ∞*-convergent sequences.*

3. *Define the metric* ρ^λ *on* $M^\lambda = S \cup \partial^\lambda M$ *as follows:*

$$\rho^\lambda(a, b) = \sum_{n=1}^{\infty} \frac{1}{2^n} \big(|\delta_{a,n} - \delta_{b,n}| + d(K^\lambda(a, n), K^\lambda(b, n)) \big),$$

   where $d(i, j) = \frac{|i-j|}{1+|i-j|}$.

4. *Let* $S^\lambda = \{[\mathbf{x}] \in \partial^\lambda M : K^\lambda([\mathbf{x}], \cdot)$ *is a QSD*$\}$.

The next theorem is a Choquet-type result, stating that the metric space introduced above characterizes all QSDs through the ways the process may "come from infinity".

Theorem 2.3. *Let* $\lambda > 0$ *be in the finite MGF regime. Then, there exists a QSD with absorption parameter* λ *if and only if* S^λ *is not empty. In this case,* μ *is a QSD with absorption parameter* λ *if and only if there exists a probability measure* $\bar{F}_\mu$ *on* $\partial^\lambda M$ *satisfying* $\bar{F}(S^\lambda) = 1$ *and*

$$\mu(y) = \int K^\lambda([\mathbf{x}], y) d\bar{F}_\mu([\mathbf{x}]), \ y \in S.$$

Analogous results in the context of positive harmonic functions are classical results in potential theory, e.g., [23] and our proof of Theorem 2.3 is primarily based on the [23] with the appropriate changes. Nevertheless, to the best of our knowledge, the present work is the first to introduce the Martin boundary in the context of QSDs and the first attempt to characterize all QSDs in the finite moment regime.

Theorems 2.1 and 2.3 provide a complete description of all QSDs for a given Markov chain.

2.2.2 Conditions for Existence of QSDs

In this section, we focus on the behavior of the generating function at infinity to show sufficient conditions for the existence and the non-existence of a QSD.

Theorem 2.4. *Let $\lambda > 0$ be in the finite MGF regime. Then*

1. *If there exists $\lambda' \in (0, \lambda)$ satisfying $\lim_{x\to\infty} E_x[\exp(\lambda'\tau_\Delta)] = \infty$ then for every λ, ∞-convergent sequence $\mathbf{x}$, $K^\lambda(\mathbf{x}, \cdot)$ is a QSD with absorption parameter λ.*

2. *If $\sup_x E_x[\exp(\lambda\tau_\Delta)] < \infty$, then there are no QSDs with absorption parameter λ.*

Corollary 2.1. *Let*

$$\lambda_0 = \inf\{\lambda \in (0, \lambda_{cr}) : \lim_{x\to\infty} E_x[\exp(\lambda\tau_\Delta)] = \infty\},$$

with the convention $\inf \emptyset = \infty$. Then for every $\lambda \in (\lambda_0, \lambda_{cr}]$ there exists a QSD with absorption parameter λ.

Note that the corollary yields the existence of a minimal QSD, regardless of whether it is in the infinite or finite MGF regime.

A tightness argument, a key element in the proof of the first part of the theorem and the corollary, has appeared in [10]. The proof of the second part of the theorem relies on results to be developed independently in the next section.

Definition 2.3. *Let $\lambda > 0$ be in the finite MGF regime and for $x, y \in S$ define*

$$C^\lambda(x, y) = \frac{E_x[\exp(\lambda\tau_\Delta), \tau_y < \tau_\Delta]}{E_x[\exp(\lambda\tau_\Delta)] - 1}.$$

With additional assumptions on p, the existence and uniqueness of QSDs can be obtained through analysis of C^λ.

Proposition 2.3. *Assume that for every $y \in S$,*

$$\sum_z p(z, y) < \infty. \tag{2.11}$$

Let $\lambda > 0$ be in the finite MGF regime. Let $(x_n : n \in \mathbb{N})$ be a sequence with $\lim_{n\to\infty} x_n = \infty$.

1. *If $\liminf_n C^\lambda(x_n, y) > 0$ for some $y \in S$, then there exists a QSD with absorption parameter λ.*

2. *If $\liminf_n C^\lambda(x_n, y) \geq 1$ for all $y \in S$, then there exists a unique QSD with absorption parameter λ given by*

$$\nu(y) = \frac{e^\lambda - 1}{E_y[\exp(\lambda\tau_\Delta), \tau_\Delta < \tau_y]}.$$

Corollary 2.2. *Assume that* (2.11) *holds. If the set* $A = \{x : p(x, \Delta) > 0\}$ *is finite, then for every* $\lambda \in (0, \lambda_{cr})$ *there exists a QSD with absorption parameter* λ.

We close this section with a result connecting the Martin boundary approach Proposition 2.3.

Corollary 2.3. *Let* $\lambda > 0$ *be in the finite MGF regime and suppose that* (2.11) *holds. Let* $[\mathbf{x}] \in \partial^\lambda M$. *Then the following are equivalent:*

1. $K^\lambda([\mathbf{x}], \cdot)$ *satisfies* (1.6) *and is not identically zero.*
2. *There exists* $y \in S$ *and a sequence* $(x_n : n \in \mathbb{N}) \in [\mathbf{x}]$ *such that* $\lim_{n\to\infty} C^\lambda(x_n, y) > 0$.

Note that under the equivalent conditions in the corollary and the irreducibility of S, $K^\lambda([\mathbf{x}], \cdot)$ is strictly positive, and $\lim_{n\to\infty} C^\lambda(x_n, y)$ exists and is in $(0, \infty)$ for all $y \in S$ and $(x_n : n \in \mathbb{N}) \in [\mathbf{x}]$. Moreover, $K^\lambda([\mathbf{x}], \cdot)$ can be normalized to a probability measure which is then necessarily a QSD.

2.3 Auxiliary Results

Proposition 2.4. *The family of distributions of* $e^{\lambda_{cr}\tau_\Delta}$ *under* P_x, $x \in S$, *is not uniformly integrable.*

The proposition has the following immediate corollary utilizing the fact that any finite set of integrable RVs are uniformly integrable and stochastically dominated.

Corollary 2.4.

1. *Suppose that* S *is finite. Then* λ_{cr} *is in the infinite regime.*
2. *Suppose that there exists a sequence* $(x_n : n \in \mathbb{N})$ *of elements in* S *and that for every* $n \in \mathbb{N}$ *the distribution of* τ_Δ *under* P_{x_n} *is stochastically dominated by its distribution under* $P_{x_{n+1}}$. *If* λ_{cr} *is in the finite regime,* $\lim_{n\to\infty} E_{x_n}[\exp(\lambda_{cr}\tau_\Delta)] = \infty$.

Proof of Proposition 2.4. We argue by contradiction. Suppose that $\lim_{n\to\infty} \sup_{x\in S} E_x[\exp(\lambda_{cr}\tau_\Delta), \tau_\Delta > n] = 0$. Then for every $\epsilon \in (0, 1)$, there exists n_0 such that $\sup_x E_x[\exp(\lambda_{cr}\tau_\Delta), \tau_\Delta > n_0] < \epsilon/2$. As a result of the Markov property, it follows that $\sup_x E_x[\exp(\lambda_{cr}\tau_\Delta), \tau_\Delta > kn_0] \le (\epsilon/2)^k$. And so, for every $x \in S$,

$$\sum_{k=0}^{\infty} \epsilon^{-k} E_x[\exp(\lambda_{cr}\tau_\Delta), \tau_\Delta > kn_0] < \infty.$$

This implies

$$\sum_{k=0}^{\infty} \epsilon^{-(k+1)} E_x[\exp(\lambda_{cr}\tau_\Delta), kn_0 < \tau_\Delta \le (k+1)n_0] < \infty$$

But when $\tau_\Delta \le (k+1)n_0$, $\epsilon^{-(k+1)} \ge \epsilon^{-\tau_\Delta/n_0}$, and so we have

$$\sum_{k=0}^{\infty} E_x[(\epsilon^{-1/n_0} e^{\lambda_{cr}})^{\tau_\Delta}, kn_0 < \tau_\Delta \le (k+1)n_0] < \infty.$$

That is, taking $e^{\widetilde{\lambda_{cr}}} = \epsilon^{-1/n_0} e^{\lambda_{cr}} > e^{\lambda_{cr}}$, we have that $E_x[e^{\widetilde{\lambda_{cr}}\tau_\Delta}] < \infty$, contradicting the definition of λ_{cr}. □

Next, we provide sufficient conditions for (2.1) to hold.

Proposition 2.5. *Let $x \in S$ and suppose that at least one of the following conditions hold:*

1. *The probability distributions $r \to P_x(X_r \in \cdot \mid \tau_\Delta \wedge \tau_x > r)$ are tight.*
2. *There exists $x \in S$ such that $\inf_{y \in S} P_y(\tau_x < \tau_\Delta) > 0$.*

Then $\sup_x E_x[\exp(\lambda_{cr}\tau_\Delta \wedge \tau_x)] < \infty$.

Note that if the set S of states z satisfying $p(z, \Delta) > 0$ is finite, then the second condition automatically holds.

Proof of Proposition 2.5. Let $\lambda \in (0, \lambda_{cr})$. Summing by parts, for any nonnegative bounded random variable Z, we have

$$E_x[\exp(\lambda\tau_\Delta \wedge \tau_x)Z] = e^\lambda E_x[Z] + (e^\lambda - 1)\sum_{r=1}^{\infty} e^{\lambda r} E_x[Z\mathbf{1}_{\{\tau_\Delta \wedge \tau_x > r\}}]. \tag{2.12}$$

By monotone convergence, this holds for any nonnegative random variable Z. When taking $Z = 1$ we have

$$E_x[\exp(\lambda\tau_\Delta \wedge \tau_x)] = e^\lambda + (e^\lambda - 1)\sum_{r=1}^{\infty} e^{\lambda r} P_x(\tau_\Delta \wedge \tau_x > r). \tag{2.13}$$

Now take

$$Z = E_{X(\tau_\Delta \wedge \tau_x)}[\exp(\lambda^0 \tau_\Delta)].$$

The strong Markov property gives that the left-hand side of (2.12) is equal to $E_x[\exp(\lambda\tau_\Delta)]$. As for the right-hand side, from the Markov property,

$$E_x[Z, \tau_\Delta \wedge \tau_x > r | \mathcal{F}_r] = \mathbf{1}_{\{\tau_\Delta \wedge \tau_x > r\}} E_{X(\tau_\Delta \wedge \tau_x)}[\exp(\lambda^0 \tau_\Delta)].$$

In our case,

$$E_x[Z, \tau_\Delta \wedge \tau_x > r | \mathcal{F}_r] \ge \mathbf{1}_{\{\tau_\Delta \wedge \tau_x > r\}} P_{X(r)}(\tau_x < \tau_\Delta) E_x[\exp(\lambda\tau_\Delta)].$$

Therefore

$$\begin{aligned} E_x[Z, \tau_\Delta \wedge \tau_x > r] &\ge E_x[\exp(\lambda\tau_\Delta)] E_x[\mathbf{1}_{\{\tau_\Delta \wedge \tau_x > r\}} P_{X(r)}(\tau_x < \tau_\Delta)] \\ &= E_x[\exp(\lambda\tau_\Delta)] E_x[P_{X(r)}(\tau_x < \tau_\Delta) | \tau_\Delta \wedge \tau_x > r] P_x(\tau_\Delta \wedge \tau_x > r). \end{aligned}$$

Assuming the first condition, the tightness condition. For every $\epsilon > 0$ there exists some finite set K_ϵ such that $P(X(r) \in K_\epsilon | \tau_\Delta \wedge \tau_x > r) \geq (1-\epsilon)$. Let $c_2 = c_2(\epsilon) = \min_{y \in K_\epsilon} P_y(\tau_x < \tau_\Delta) > 0$. Therefore, we have that

$$E_x[P_{X(r)}(\tau_x < \tau_\Delta) | \tau_\Delta \wedge \tau_x > r] \geq c_1, \tag{2.14}$$

where $c_1 = (1-\epsilon)c_2$. If we assume the second condition instead, then we can use the infimum in the condition as c_1 in (2.14). Thus, under either condition, we have

$$\begin{aligned} E_x[\exp(\lambda\tau_\Delta)] &\geq e^\lambda E_x[E_{X(\tau_\Delta \wedge \tau_x)}[\exp(\lambda^0 \tau_\Delta)]] + c_1 E_x[\exp(\lambda\tau_\Delta)](e^\lambda - 1)\sum_{r=1}^{\infty} e^{\lambda r} P_x(\tau_\Delta \wedge \tau_x > r) \\ &\overset{(2.13)}{=} e^\lambda E_x[E_{X(\tau_\Delta \wedge \tau_x)}[\exp(\lambda^0 \tau_\Delta)]] + c_1 E_x[\exp(\lambda\tau_\Delta)](E_x[\exp(\lambda\tau_\Delta \wedge \tau_x)] - e^\lambda) \\ &\geq c_1 E_x[\exp(\lambda\tau_\Delta)](E_x[\exp(\lambda\tau_\Delta \wedge \tau_x)] - e^\lambda). \end{aligned}$$

Divide both sides by $E_x[\exp(\lambda\tau_\Delta)]$ to obtain the bound

$$E_x[\exp(\lambda\tau_\Delta \wedge \tau_x)] \leq e^\lambda + \frac{1}{c_1}.$$

The result now follows from monotone convergence. □

3 Proof of the results of Section 2.1

We begin with some results we need.

3.1 Potential Theoretic Results

Proposition 3.1. *Let $\lambda \in (0, \lambda_{cr}]$.*

1. *For $x \in S$ define the measure μ_x through*

$$\begin{aligned} \mu_x(y) &= E_x[\sum_{s < \tau_x \wedge \tau_\Delta} \exp(\lambda s)\delta_y(X_s)],\ y \in S \\ &= \frac{E_x[\exp(\lambda\tau_y), \tau_y < \tau_x \wedge \tau_\Delta]}{1 - E_y[\exp(\lambda\tau_y), \tau_y < \tau_x \wedge \tau_\Delta]} \end{aligned} \tag{3.1}$$

   Then

$$(\mu_x p)(z) = e^{-\lambda}\mu_x(z) + e^{-\lambda}\delta_x(z)\left(E_x[\exp(\lambda\tau_x), \tau_x < \tau_\Delta] - 1\right),\ z \in S. \tag{3.2}$$

2. *For $z \in S$ define the function $h_z : S \to [0, \infty)$ by letting*

$$h_z(x) = E_x[\exp(\lambda\tau_z), \tau_z < \tau_\Delta] \tag{3.3}$$

   Then

$$(ph_z)(x) = e^{-\lambda}h_z(x) + p(x,z)(h_z(z) - 1) \tag{3.4}$$

Note that Proposition 1.2 and the irreducibility guarantee that both ν_x and h_z defined in the proposition are strictly positive and finite on S.

Proof. For the first claim,

$$\begin{aligned}
\mu_x p(z) &= \sum_y \sum_{s=0}^{\infty} E_x[\mathbf{1}_{\tau_\Delta > s}\mathbf{1}_{\tau_x > s} \exp(\lambda s)\delta_y(X_s)]p(y,z) \\
&= e^{-\lambda} \sum_{s=0}^{\infty} e^{\lambda(s+1)} E_x[\mathbf{1}_{\tau_\Delta > s}\mathbf{1}_{\tau_x > s}\delta_z(X_{s+1})] \\
&= e^{-\lambda} \left(\mu_x(z) - \delta_x(z) + \delta_x(z) E_x[\exp(\lambda\tau_x), \tau_x < \tau_\Delta]\right) \\
&= e^{-\lambda}\mu_x(z) + e^{-\lambda}\delta_x(z) \left(E_x[\exp(\lambda\tau_x), \tau_x < \tau_\Delta] - 1\right).
\end{aligned}$$

For the second claim, observe that

$$\begin{aligned}
h_z(x) &= E_x[\exp(\lambda\tau_z), \tau_z < \tau_\Delta] \\
&= e^{\lambda} p(x,z) + \sum_{y \neq z} e^{\lambda} p(x,y) E_y[\exp(\lambda\tau_z), \tau_z < \tau_\Delta] \\
&= e^{\lambda} \sum_{y \in S} p(x,y) h_z(y) + e^{\lambda} p(x,z)(1 - h_z(z)).
\end{aligned}$$

□

3.2 The Reverse Chain

Suppose that ν is a QSD with absorption parameter λ. We introduce the time-reversed transition function q on S:

$$q(y,x) = \nu(x)p(x,y)\frac{e^{\lambda}}{\nu(y)}, \ x, y \in S. \tag{3.5}$$

Note that q has no absorbing states, inherits the irreducibility from p, and reverses the arrow of time. Write $Q_\cdot, E_\cdot^Q$ for the probability and expectation for the Markov Chain on S corresponding to the transition function q. We have the following simple lemma obtained from products of (3.5).

Lemma 3.1. *Let* $\mathbf{x} = (x_0, x_1, \ldots, x_n)$ *be a sequence in* S. *Write* $\overleftarrow{\mathbf{x}} = (x_n, x_{n-1}, \ldots, x_0)$, *the reverse sequence. Then*

$$\prod_{j=0}^{n-1} p(x_j, x_{j+1}) = e^{-\lambda n} \frac{\nu(x_n)}{\nu(x_0)} \prod_{j=n}^{1} q(x_j, x_{j-1}).$$

In particular

$$P_{x_0}(X_{[0,n]} = \mathbf{x}) = e^{-\lambda n} \frac{\nu(x_n)}{\nu(x_0)} Q_{x_n}(X_{[0,n]} = \overleftarrow{\mathbf{x}}), \tag{3.6}$$

and

$$P_{x_0}(X_{[0,n]} = \mathbf{x}, \tau_\Delta = n+1) = e^{-\lambda n} \frac{\nu(x_n)p(x_n,0)}{\nu(x_0)} Q_{x_n}(X_{[0,n]} = \overleftarrow{\mathbf{x}}). \tag{3.7}$$

To introduce the next result, we need some additional notations. Let

$$I_\nu(x) = e^\lambda \sum_z \nu(z)p(z,\Delta)Q_z({}^0\tau_x < \infty). \tag{3.8}$$

Proposition 3.2. *Let ν be a QSD for p with absorption parameter λ. Then $I_\nu(x) \le e^\lambda - 1$ and equality holds if and only if one of the following holds:*

1. *q is recurrent.*
2. *There's a bijection $\sigma : S \to \mathbb{Z}_+ \cup \{-1\}$ with $\sigma(\Delta) = -1$ and the following properties*
   (a) *For all $x \in S$, $p(x,y) > 0$ if $\sigma(y) = \sigma(x) - 1$.*
   (b) *For all $x \in S$, $p(x,y) = 0$ if $\sigma(y) < \sigma(x) - 1$.*

A chain satisfying the latter set of conditions is also known as skip-free [11]. Such chains are the simplest to study. Complete characterization of all QSDs for such chains is given in Section 6.1.

Proof. Since $\nu(S) = 1$, we have

$$I_\nu(x) \le e^\lambda \sum_z \nu(z)p(z,\Delta) = e^\lambda \sum_z \nu(z)(1 - \sum_{y\neq\Delta} p(z,y)) = e^\lambda(1-e^{-\lambda}) = e^\lambda - 1. \tag{3.9}$$

The inequality in (3.9) is an equality if and only if for every z with $p(z,\Delta) > 0$, $Q_z({}^0\tau_x < \infty) = 1$. This clearly holds if q is recurrent or if item 2 in the statement of the Proposition 3.2 holds. Conversely, if neither occurs, then there exists $x, z \in S$ with $p(z,\Delta) > 0$ such that $Q_z({}^0\tau_x < \infty) < 1$. □

As an application of Lemma 3.1, we have

Proposition 3.3. *Let ν be a QSD for p with absorption parameter λ. Then*

1. $E_x[\exp(\lambda\tau_x), \tau_x < \tau_\Delta] = Q_x(\tau_x < \infty)$.
2. $E_x[\exp(\lambda\tau_\Delta), \tau_\Delta < \tau_x] = \frac{I_\nu(x)}{\nu(x)}$.
3. $E_x[\exp(\lambda\tau_\Delta)] = \frac{I_\nu(x)}{\nu(x)} E_x^Q[N(x)]$ *where,* $N(x) = \sum_{s=0}^\infty \delta_x(X_s)$.
4. $E_x[\exp(\lambda\tau_x)\tau_x, \tau_x < \tau_\Delta] = E_x^Q[\tau_x, \tau_x < \infty]$.

Note that under the assumptions of the proposition, q is recurrent if and only if $E_x[\exp(\lambda_{cr}\tau_x), \tau_x < \tau_\Delta] = 1$ for some $x \in S$, and if this equality holds, it is positive recurrent if and only if $E_x[\exp(\lambda_{cr}\tau_x)\tau_x, \tau_x < \tau_\Delta] < \infty$ for some $x \in S$ with a stationary distribution $\pi(x) = \frac{1}{E_x^Q[\tau_x]} = \frac{1}{E_x[\exp(\lambda_{cr}\tau_x)\tau_x, \tau_x < \tau_\Delta]}$ for all $x \in S$.

Proof. For $n \in \mathbb{Z}_+$, $x \in S$, let $A_x(n)$ be the set of paths $\mathbf{x} = (x_0, x_1, \cdots, x_n)$ with $x_0 = x$. Also, let $A_x^-(n) \subset A_x(n)$ the subset of paths satisfying $x_1, \ldots, x_{n-1} \neq x$, and finally, let $A_{x,x}^-(n)$ be the subset of $A_x^-(n)$ consisting of paths satisfying $x_n = x$.

For the first assertion, using (3.5) and (3.6), for $0 < i < n$ we have

$$\begin{aligned} E_x[\exp(\lambda\tau_x), \tau_x < \tau_\Delta] &= \sum_{n=0}^{\infty} \sum_{\mathbf{x} \in A_{x,x}^-(n)} e^{\lambda n} P_x(X_{[0,n]} = \mathbf{x}) \\ &= \sum_{n=0}^{\infty} \sum_{\mathbf{x} \in A_{x,x}^-(n)} Q_x(X_{[0,n]} = \overleftarrow{\mathbf{x}}) \\ &= Q_x(\tau_x < \infty) \end{aligned}$$

For the second assertion, consider $z \in S$. Using (3.7), we obtain

$$\begin{aligned} E_x[\exp(\lambda\tau_\Delta), \tau_\Delta < \tau_x] &= \sum_{n=0}^{\infty} \sum_{\mathbf{x} \in A_x^-(n)} e^{\lambda(n+1)} P_x(X_{[0,n]} = \mathbf{x}, \tau_\Delta = n+1) \\ &= \sum_{z \in S} \frac{e^\lambda}{\nu(x)} \nu(z) p(z, \Delta) Q_z({}^0\tau_x < \infty) \\ &= \frac{I_\nu(x)}{\nu(x)} \end{aligned}$$

For the third assertion, using (3.7) for $z \in S$

$$\begin{aligned} E_x[\exp(\lambda\tau_\Delta)] &= \sum_{n=0}^{\infty} \sum_{\mathbf{x} \in A_x(n)} e^{\lambda(n+1)} P_x(X_{[0,n]} = \mathbf{x}, \tau_\Delta = n+1) \\ &= \sum_{z \in S} \frac{e^\lambda}{\nu(x)} \nu(z) p(z, \Delta) Q_z(x_0 = x) \\ &= \sum_{z \in S} \frac{e^\lambda}{\nu(x)} \nu(z) p(z, \Delta) Q_z({}^0\tau_x < \infty) E_x^Q[N(x)] \\ &= \frac{I_\nu(x)}{\nu(x)} E_x^Q[N(x)] \end{aligned}$$

To prove the last assertion, using (3.6)

$$\begin{aligned}
E_x^Q[\tau_x, \tau_x < \infty] &= \sum_{n=1}^{\infty} n \sum_{\mathbf{x} \in A_{x,x}^-(n)} Q_x(X_{[0,n]} = \mathbf{x}) \\
&= \sum_{n=1}^{\infty} n \sum_{\mathbf{x} \in A_{x,x}^-(n)} e^{\lambda n} P_x(X_{[0,n]} = \mathbf{x}) \\
&= \sum_{n=1}^{\infty} n e^{\lambda n} \sum_{\mathbf{x} \in A_{x,x}^-(n)} P_x(X_{[0,n]} = \mathbf{x}) \\
&= E_x[\tau_x \exp(\lambda \tau_x), \tau_x < \tau_\Delta]
\end{aligned}$$

□

Proposition 3.2 and Proposition 3.3-2 give the following:

Corollary 3.1. *Let ν be a QSD with absorption parameter λ. Then for all $x \in S$*

$$\nu(x) = \frac{I_\nu(x)}{E_x[\exp(\lambda \tau_\Delta), \tau_\Delta < \tau_x]} \leq \frac{e^\lambda - 1}{E_x[\exp(\lambda \tau_\Delta), \tau_\Delta < \tau_x]}. \tag{3.10}$$

The transition function q may be useful in the study of convergence to a specific QSD. Indeed, if ν is a QSD with absorption parameter λ and q is the transition function for the reverse process as defined in (3.5), then for any probability measure μ on S and any $y \in S$

$$P_\mu(X_n = y) = e^{-\lambda n} \nu(y) E_y^Q[\frac{\mu}{\nu}(X_n)].$$

This implies

$$P_\mu(X_n = y | \tau_\Delta > n) = \nu(y) \frac{E_y^Q[\frac{\mu}{\nu}(X_n)]}{E_\nu^Q[\frac{\mu}{\nu}(X_n)]} \tag{3.11}$$

Corollary 3.2. *Suppose that μ is a probability measure satisfying $\sup_{y \in S} \frac{\mu}{\nu}(y) < \infty$. Then, each of the conditions below implies*

$$\lim_{n \to \infty} P_\mu(X_n \in \cdot \mid \tau_\Delta > n) = \nu(y).$$

1. *q is positive recurrent and aperiodic.*
2. *q is transient and $\lim_{y \to \infty} \frac{\nu}{\mu}(y)$ exists and is strictly positive.*

Of course, both parts of the corollary follow from (3.10). The first part is a straightforward application of the ergodic theorem for positive recurrent and aperiodic Markov chains, e.g., [29, Chapter 3], and the second is a trivial application of the transience assumption.

3.3 Proof of Theorem 2.1

Uniqueness and necessity. Suppose first that ν is a minimal QSD. We construct the reverse chain corresponding to (3.5). Proposition 3.3-3 gives that for every $x \in S$, $\infty = E_x[\exp(\lambda\tau_\Delta)] = \frac{I_\nu(x)}{\nu(x)} E_x^Q[N(x)]$. The ratio on the righthand side is finite due to (3.9), and so $E_x^Q[N(x)]$, the expected number of visits to x by the reverse chain, is infinite. As a result, q is recurrent. Proposition 3.2-1 then gives $I_\nu(x) = e^{\lambda_{cr}} - 1$, and then Proposition 3.3-2 gives the representation (2.2) for ν. This proves the uniqueness of a minimal QSD and also implies (2.1) due to Proposition 1.2.

Existence and sufficiency. Suppose that (2.1) holds. Then by Proposition 1.2, $E_x[\exp(\lambda_{cr}\tau_x), \tau_x < \tau_\Delta] = 1$, and therefore the measure μ_x from Proposition 3.1 with $\lambda = \lambda_{cr}$ satisfies (1.6). Since $\mu_x(S) = \dfrac{E_x[\exp(\lambda_{cr}\tau_\Delta \wedge \tau_x)] - 1}{e^{\lambda_{cr}} - 1}$, μ_x can be normalized to a probability measure we denote by $\bar{\mu}_x$. Proposition 1.1 implies that $\bar{\mu}_x$ is a minimal QSD.

Convergence. Since q is already recurrent, the additional assumption (2.3) and Proposition 1.2-4 give that q is positive recurrent. And then, we apply the first part of Corollary 3.2. □

3.4 Proof of Proposition 2.1

The statement is trivial when $|K| = 1$. We will show that if $|K| \geq 2$, there exists $x_0 \in K$ so that

$$E_{x_0}[\exp(\lambda_{cr}\tau_\Delta \wedge \tau_{K-\{x_0\}})] < \infty, \tag{3.12}$$

and so by iterating, we can eventually reduce to the case $|K| = 1$. We, therefore, turn to prove (3.12). For $\lambda < \lambda_{cr}$ and $x \in S$,

$$\begin{aligned}\infty > E_x[\exp(\lambda\tau_\Delta)] = {} & E_x[\exp(\lambda\tau_\Delta \wedge \tau_K)\mathbf{1}_{\{\tau_K < \tau_\Delta\}}] E_{X(\tau_K)}[\exp(\lambda\tau_\Delta)] \\ & + E_x[\exp(\lambda\tau_\Delta \wedge \tau_K)\mathbf{1}_{\{\tau_\Delta < \tau_K\}}].\end{aligned}$$

Let $v_m(\lambda)$ denote the minimum of the lefthand side over $x \in K$, and let $x = x_m \in K$ be a minimizer. Then,

$$v_m(\lambda) \geq E_{x_m}[\exp(\lambda\tau_\Delta \wedge \tau_K)\mathbf{1}_{\{\tau_K < \tau_\Delta\}}] v_m(\lambda),$$

and therefore $E_{x_m}[\exp(\lambda\tau_\Delta \wedge \tau_K)\mathbf{1}_{\{\tau_K < \tau_\Delta\}}] \leq 1$. Let $\lambda \nearrow \lambda_{cr}$ along any sequence $(\lambda_n : n \in \mathbb{N})$. Then there is a subsequence, which we also denote by $(\lambda_n : n \in \mathbb{N})$ so that $\nu_m(\lambda_n)$ is constant. The Monotone Convergence Theorem then yields that for some $x_m \in K$,

$$E_{x_m}[\exp(\lambda_{cr}\tau_\Delta \wedge \tau_K)\mathbf{1}_{\{\tau_K < \tau_\Delta\}}] \leq 1.$$

Note that this is a version of Proposition 1.2 but with the singleton replaced by a finite set. Consider any shortest path from x_m to $x \in K - \{x_m\}$ which does

not return to x_m. Since each such path has a positive probability under P_{x_m}, it follows that $P_{x_m}(\tau_{x_m} = \tau_K, \tau_K < \tau_\Delta) < P_{x_m}(\tau_K < \tau_\Delta)$. With this, we have

$$\rho_m = E_{x_m}[\exp(\lambda_{cr}\tau_\Delta \wedge \tau_K)\mathbf{1}_{\{\tau_{x_m}=\tau_\Delta\wedge\tau_K\}}] < 1.$$

For $\lambda < \lambda_{cr}$ the strong Markov property gives

$$\begin{aligned}\infty > E_{x_m}[\exp(\lambda\tau_\Delta \wedge \tau_{K-\{x_m\}})] &= \rho_m E_{x_m}[\exp(\lambda\tau_\Delta \wedge \tau_{K-\{x_m\}})] \\ &\quad + E_{x_m}[\exp(\lambda\tau_\Delta \wedge \tau_K)\mathbf{1}_{\{\tau_{x_m}>\tau_\Delta\wedge\tau_K}\}].\end{aligned}$$

Therefore

$$\begin{aligned}E_{x_m}[\exp(\lambda\tau_\Delta \wedge \tau_{K-\{x_m\}})] &\le \frac{E_{x_m}[\exp(\lambda\tau_\Delta \wedge \tau_K)\mathbf{1}_{\{\tau_{x_m}>\tau_\Delta\wedge\tau_K}\}]}{1-\rho_m} \\ &\le \frac{E_{x_m}[\exp(\lambda_{cr}\tau_\Delta \wedge \tau_K)]}{1-\rho_m}.\end{aligned}$$

The result follows by applying the monotone convergence theorem on the left-hand side. □

3.5 Proof of Theorem 2.2 and Proposition 2.2

We begin by presenting a sufficient condition for λ_{cr} to be in the infinite MGF regime.

Proposition 3.4. *Suppose that $K \subsetneq S$ is nonempty and finite and that for some $\bar{\lambda} > \lambda_{cr}$*

$$\sup_x E_x[\exp(\bar{\lambda}\tau_\Delta \wedge \tau_K)] < \infty. \tag{3.13}$$

Then λ_{cr} is in the infinite MGF regime.

Proof. We argue by contradiction assuming $E_x[\exp(\lambda_{cr}\tau_\Delta)] < \infty$ for all $x \in S$. For $x \in S$, let $T_1(x)$ be a random variable whose distribution is the same as $\tau_\Delta \wedge \tau_K$ under P_x and let T_2 be independent of $T_1(x)$ and equal to the sum of $|K|$, independent random variables $(T_{1,k} : k \in K)$ with $T_{1,k}$ are distributed according to τ_Δ under P_k. Then the distribution of τ_Δ under P_x is stochastically dominated by $T_1(x) + T_2$. Therefore

$$\begin{aligned}E_x[\exp(\lambda_{cr}\tau_\Delta)\mathbf{1}_{\tau_\Delta>2n}] &\le E[\exp(\lambda_{cr}(T_1(x)+T_2)), T_1(x)+T_2 > 2n] \\ &\le E[\exp(\lambda_{cr}(T_1(x)+T_2))(\mathbf{1}_{\{T_1(x)>n\}} + \mathbf{1}_{\{T_2>n\}})] \\ &\le E[\exp(\lambda_{cr}T_2)]E_x[\exp(\lambda_{cr}T_1(x)), T_1(x) > n] \\ &\quad + E[\exp(\lambda_{cr}T_1(x))]E_x[\exp(\lambda_{cr}T_2), T_2 > n] \\ &= (*),\end{aligned}$$

with all expectations on the righthand side being finite. Let $\delta = \bar{\lambda} - \lambda_{cr}$. Then on the event $\{T_1(x) > n\}$

$$\exp(\lambda_{cr}T_1(x)) = \exp(\bar{\lambda}T_1(x))\exp(-\delta T_1(x)) \le \exp(\bar{\lambda}T_1(x))e^{-\delta n}.$$

Therefore

$$E[\exp(\lambda_{cr}T_1(x)), T_1(x) > n] \leq E[\exp(\bar{\lambda}T_1(x))]e^{-\delta n} \leq c_1 e^{-\delta n},$$

where $c_1 < \infty$ is the supremum in (3.13). Then,

$$(*) \leq E[\exp(\lambda_{cr}T_2)]c_1 e^{-\delta n} + c_1 E[\exp(\lambda_{cr}T_2), T_2 > n].$$

This upper bound is independent of x and tends to 0 as $n \to \infty$. Thus the the distributions of $e^{\lambda_{cr}\tau_\Delta}$ under P_x as x ranges over S is uniformly integrable, which is in contradiction to Proposition 2.4. □

Proof of Theorem 2.2. We first show that λ_{cr} is in the infinite MGF regime. From (2.6) we learn that $\lambda_{cr} \leq \bar{\lambda}$. If an equality holds, then λ_{cr} is in the infinite MGF regime. Otherwise, $\lambda_{cr} < \bar{\lambda}$, and the claim follows from Proposition 3.4.

With this, Proposition 2.1 guarantees that both conditions of Theorem 2.1 hold and therefore, there exists a unique minimal QSD ν, given by (2.2). However, the assumptions yield more.

Let $z \in K$ be as in Proposition 2.1, and define $h(x) = h_z(x) = E_x[\exp(\lambda_{cr}\tau_z), \tau_z < \tau_\Delta]$. Proposition 1.2 implies that $h(z) = 1$, and therefore Proposition 3.1 gives

$$ph = e^{-\lambda_{cr}}h. \tag{3.14}$$

We show that h is also bounded. Indeed, for every $x \in S$,

$$h(x) \leq E_x[\exp(\lambda_{cr}\tau_K), \tau_K < \tau_\Delta] \max_{x\in K} h(x) < \infty.$$

Therefore, νh can be normalized to be a probability measure. Let $\bar{h} = h/(\sum_x \nu h)$ and let $\pi = \nu\bar{h}$ be a probability measure. Let p^h be the kernel on S defined by

$$p^h(x,y) = \frac{e^{\lambda_{cr}}}{h(x)}p(x,y)h(y).$$

Then (3.14) guarantees that p^h is a transition function on S. Moreover,

$$\pi p^h = e^{\lambda_{cr}}\nu\frac{\bar{h}}{h}ph = \nu\bar{h} = \pi,$$

and therefore p is positive recurrent and π is its stationary distribution. A similar calculation shows that q is positive recurrent, and therefore Proposition 3.3-4 guarantees that the condition (2.3) holds.

To show that no other QSD exists, note that if ν' is a QSD with survival parameter $\lambda \leq \lambda_{cr}$, then

$$\sum_{x,y}(\nu' h)(x)p^h(x,y) = \sum_x \nu'(x)p(x,y)h(y),$$

and the sum is finite because $\nu' h$ is a finite measure. Summing over x first yields $e^{-\lambda}\sum_x \nu'(y)h(y)$, while summing over y first yields $e^{-\lambda_{cr}}\sum_x \nu'(x)h(x)$. Therefore, $\lambda = \lambda_{cr}$ and the claim follows.

It remains to prove the convergence result. From the construction of p^h we have that for any initial distribution μ,

$$P_\mu(X_n = y|\tau_\Delta > n) = \frac{\sum_{x\in S}\mu(x)h(x)P_x^h(X_n = y)/h(y)}{\sum_{x\in S}\mu(x)h(x)E_x^h[\frac{1}{h(X_n)}]} = \frac{1}{h(y)}\frac{P_{\bar\mu}^h(X_n = y)}{E_{\bar\mu}^h[\frac{1}{h(X_n)}]}, \tag{3.15}$$

where $\bar\mu$ is the probability measure and $\bar\mu(x) = (\mu h)(x)/\sum_y(\mu h)(y)$.

Next, from (3.14) we conclude that $(e^{\lambda_{cr}n}h(X_n)\mathbf{1}_{\{\tau_\Delta>n\}} : n \in \mathbb{Z}_+)$ is a martingale. In particular, for every $n \in \mathbb{Z}_+$,

$$h(x) = E_x[\exp(\lambda_{cr}\tau_{x_0} \wedge n)h(X_{\tau_{x_0}\wedge n}), \tau_{x_0} \wedge n < \tau_\Delta].$$

As h is bounded, and so is the mapping $x \to E_x[\exp(\lambda_{cr}\tau_{x_0}), \tau_{x_0} < \tau_\Delta]$ (this function is equal to 1 at x_0 because q is recurrent, and is bounded by the same argument leading to the boundedness of h), it follows from the monotone convergence theorem that we can take $n \to \infty$ and obtain

$$h(x) = E_x[\exp(\lambda_{cr}\tau_{x_0}), \tau_{x_0} < \tau_\Delta]h(x_0).$$

In particular, $h(x) \geq P_x(\tau_{x_0} < \tau_\Delta)h(x_0)$, and then the condition (2.8) implies that for some constant $c > 0$, depending only on x_0 and n_0, $h(x) \geq cP_x(\tau_\Delta > n_0)$ for all $x \in S$. Equivalently, $l(x) = E_x^h[\frac{1}{h(X_{n_0})}] \leq c^{-1}$. Because this function is bounded, we can use the Markov property and the ergodic theorem for positive recurrent aperiodic Markov chains to conclude that

$$\lim_{n\to\infty} E_{\bar\mu}^h[\frac{1}{h(X_n)}] = \lim_{n\to\infty} E_{\bar\mu}^h[l(X_{n-n_0})] = \sum_x \pi(x)l(x).$$

By iterating the stationarity of π, the righthand side is equal to $\sum_x \pi(x)/h(x) = \sum_x \nu(x)\bar h(x)/h(x) = \bar h(y)/h(y)$ (the righthand side is a constant independent of y). The ergodic theorem also gives $\lim_{n\to\infty} P_{\bar\mu}^h(X_n = y) = \pi(y)$. Therefore, the limit as $n \to \infty$ of the righthand side of (3.15) is $\frac{\pi(y)}{h(y)\bar h(y)/h(y)} = \nu(y)$, completing the proof. □

Proof of Proposition 2.2. We omit the trivial case $|S| = 1$ and assume $|S| \geq 2$.

Since S is finite and irreducible, (2.8) automatically holds for every n and $x_0 \in S$.

Because any finite set of integrable RVs is uniformly integrable, Proposition 2.4 implies $E_x[\exp(\lambda_{cr}\tau_\Delta)] = \infty$ for some $x \in S$ (and therefore for all $x \in S$). This gives (2.6).

Next, since S is irreducible, there exists $y_0 \in S$ different from x_0 such that $p(x_0, y_0) > 0$. In particular $1 - p(x_0, \Delta) \geq p(x_0, x_0) + p(x_0, y_0) > p(x_0, x_0)$. The choice of x guarantees $P_x(\tau_\Delta > n) \geq (1 - p(x_0, \Delta))^n \geq (p(x_0, x_0) + p(x_0, y_0))^n$. Since for any $\lambda < \lambda_{cr}$, $\sum_{n=0}^\infty e^{\lambda n}P_x(\tau_\Delta > n)$ is finite, it follows that $e^{\lambda_{cr}}(p(x_0, x_0) + p(x_0, y_0)) \leq 1$, and in particular, $p(x_0, x_0)e^{\lambda_{cr}} < 1$. Moreover,

$$P_{x_0}(\tau_\Delta \wedge \tau_K > n) = p(x_0, x_0)^n,$$

and because $p(x_0, x_0)e^{\lambda_{cr}} < 1$, the series $\sum_{n=0}^{\infty} e^{\lambda_{cr} n} P_{x_0}(\tau_\Delta \wedge \tau_K > n)$ converges, which gives $E_{x_0}[\exp(\lambda_{cr}\tau_\Delta \wedge \tau_K)] < \infty$. Thus, (2.7) holds. □

4 Proof of the Results of Section 2.2.1

Classically, Martin Boundary theory provides a compactification of the state space of a transient Markov Chain through a set of positive harmonic functions. These functions describe the tail of the chain: under the new topology, the chain converges almost surely, with the limit viewed as where the process "exits" the state space. The books by Woess [29] and by Kemeny, Snell, and Knapp [16] are good sources for additional details on Martin boundary theory. In our work, we borrow the ideas and first construct a similar compactification of the state space, then apply analysis similar to the work by Sawyer [23].

Proposition 4.1. *Let $\lambda > 0$ be in the finite MGF regime. Let M^λ and ρ^λ be as in Definition 2.2. Then $(M^\lambda, \rho^\lambda)$ is a compact metric space.*

Proof. It is enough to show that every sequence $(x_n : n \in \mathbb{N})$ in this space has a convergent subsequence. Indeed, either

1. Some elements in S or in $\partial^\lambda M$ appear infinitely often; or
2. Every element in S appears finitely often, and so does every element in $\partial^\lambda M$. Here, at least one of the two alternatives holds:
   (a) There exists a subsequence we also denote by $(x_n : n \in \mathbb{N})$ consisting entirely of elements in S and satisfying $\lim_{n\to\infty} x_n = \infty$.
   (b) There exists a subsequenence we also denote by $(x_n : n \in \mathbb{N})$ consisting entirely of elements in $\partial^\lambda M$.

In case 1, we have a constant subsequence with an obvious limit. In case 2, we proceed according to the subcases. The case 2-(a) clearly has a λ, ∞-convergent subsequence with a limit in $\partial^\lambda M$. To treat 2-(b), for each n, let $\bar{x}_n$ be an element in S satisfying $\rho^\lambda(\bar{x}_n, x_n) < 2^{-n}$. Without loss of generality, we may assume $\lim_{n\to\infty} \bar{x}_n = \infty$. Clearly it has a convergent subsequence $(\bar{x}_{n_k} : k \in \mathbb{N})$ with limit $[\mathbf{x}]$ in $\partial^\lambda M$. By the triangle inequality, $\rho^\lambda(x_{n_k}, [\mathbf{x}]) \to 0$. □

Proof of Theorem 2.3. Let $\mathcal{M}^\lambda = \{\mu : S \to [0, \infty) : \mu p \le e^{-\lambda}\mu\}$. For any $\mu \in \mathcal{M}^\lambda$ and $n \in \mathbb{N}$

$$\mu = \mu - \mu(e^\lambda p)^n + \mu(e^\lambda p)^n.$$

The difference on the right is equal to $\mu(I - e^\lambda p)(I + (e^\lambda p) + \cdots + (e^\lambda p)^{n-1})$ and therefore increases pointwise as $n \to \infty$ to

$$\sum_x (\mu(I - e^\lambda p))(x) G^\lambda(x, y) = \sum_x (\mu(I - e^\lambda p))(x) G^\lambda(x, \mathbf{1}) K^\lambda(x, y).$$

The second term decreases pointwise to a limit we denote by μ_∞, which satisfies $\mu_\infty(e^\lambda p) = \mu_\infty$. Thus, letting $f_\mu(x) = \mu(I - e^\lambda p)(x)G^\lambda(x, \mathbf{1})$ we have

$$\mu(y) = \sum_{x\in\mathbb{N}} f_\mu(x)K^\lambda(x, y) + \mu_\infty(y),\ \ y \in S.$$

We say that $\mu \in \mathcal{M}^\lambda$ is a potential if it has a representation of the form

$$\mu(y) = \sum_{x\in S} f_\mu(x)K^\lambda(x, y)$$

for some nonnegative f_μ which is not identically zero. Thus, we proved that every $\mu \in \mathcal{M}^\lambda$ is a sum of a potential and a function μ_∞ satisfying $\sum_{y\in S} \mu_\infty(y) \in [0, \infty]$.

The first sum is an integral with respect to a Borel measure on the compact metric space $S \cup \partial^\lambda M$:

$$\mu(y) = \int_{S\cup\partial^\lambda M} K^\lambda(x, y)dF_\mu(x).$$

Where F_μ is absolutely continuous with respect to the counting measure on $\mathbb{N}$. Since $K^\lambda(x, \mathbf{1}) = 1$ for every $x \in S$, it follows that $F_\mu(\mathbf{1}) = \mu(\mathbf{1})$.

Let $\mu \in \mathcal{M}^\lambda$. If $\mu(e^\lambda p)^n \to 0$, then $\mu_\infty = 0$ and μ is a potential. Conversely, if μ is a potential, then

$$\mu(y) = \sum_{t=0}^{\infty}(g_\mu(e^\lambda p)^t)(y),$$

a convergent series. For every $n \in \mathbb{N}$, $(\mu(e^\lambda p)^n)(y) = \sum_{t=n}^{\infty}(g_\mu(e^\lambda p)^t)(y)$ which decreases pointwise to 0 as $n \to \infty$, as it is the tail of a convergent series.

Next, let $\mu, \mu' \in M^\lambda$. Then $(\mu \wedge \mu')p \le (\mu p) \wedge (\mu' p) \le e^{-\lambda}(\mu \wedge \mu')$ and so $\mu \wedge \mu' \in \mathcal{M}^\lambda$. Moreover, if μ' is a potential, it follows that

$$(\mu \wedge \mu')(e^\lambda p)^n \le \mu'(e^\lambda p)^n \to 0,$$

and so $\mu \wedge \mu'$ is a potential.

Next, let μ be a QSD with absorption parameter λ. Let $D_n = \{1, \dots, n\}$, and let $\mu_n = \mu \wedge K^\lambda(n\mathbf{1}_{D_n}, \cdot)$. Clearly, $\mu_n \nearrow \mu$, and μ_n is a potential. Therefore,

$$\mu_n(y) = \int_{S\cup\partial^\lambda M} K^\lambda(x, y)dF_n(x),$$

for some Borel measure F_n on M^λ supported on $\mathbb{N}$. Also, $F_n(\mathbf{1}) = \mu_n(\mathbf{1}) \nearrow \mu(\mathbf{1}) = 1$. Without loss of generality we may assume $F_n(\mathbf{1}) > 0$, and therefore normalize F_n to be a probability measure by letting $\bar{F}_n = F_n/F_n(\mathbf{1})$,

$$\mu_n = F_n(\mathbf{1}) \int_{S\cup\partial^\lambda M} K^\lambda(x, y)d\bar{F}_n(x).$$

As Borel probability measures on a compact metric space, $(\bar{F}_n : n \in N)$ contains a weakly convergent subsequence. Let $\bar{F}$ denote the limit. Then we have

$$\mu(y) = \int_{S \cup \partial^\lambda M} K^\lambda(x, y) d\bar{F}(x).$$

Since μ and $\bar{F}$ are both probability measures,

$$1 = \mu(\mathbf{1}) = \int K^\lambda(x, \mathbf{1}) d\bar{F}(x) \leq \bar{F}(M^\lambda) = 1.$$

Since $K^\lambda(x, \mathbf{1}) \leq 1$ for all $x \in M^\lambda$, we have that

$$\bar{F}(\{x : K^\lambda(x, \mathbf{1}) < 1\}) = 0.$$

Next, since

$$0 = \mu - \mu(e^\lambda p) = \int_{S \cup \partial^\lambda M} (K^\lambda(x, \cdot)(I - e^\lambda p))(y) d\bar{F}(x).$$

and so,

$$\bar{F}(\{x : (K^\lambda(x, \cdot)(I - e^\lambda p))(y) > 0 \text{ for some } y \in S\}) = 0.$$

Since by definition for $x \in S$, $K^\lambda(x, \cdot)(I - e^\lambda p))(x) > 0$ it follows that

$$\bar{F}(\left\{[\mathbf{x}] \in \partial^\lambda M : K^\lambda([\mathbf{x}], \cdot) \text{ is a QSD}\right\}) = 1.$$

□

5 Proof of the Results of Section 2.2.2

5.1 Preliminary Results

We now present a number of results that culminate the proof of the Theorem 2.4. Suppose $\lambda > 0$ in the finite MGF regime for Proposition 5.1, Corollary 5.1, and Proposition 5.2.

Proposition 5.1. *Fix $x \in S$. Then*

1. *For every $y \in S$,*

$$(K^\lambda(x, \cdot)p)(y) = e^{-\lambda} \left(K^\lambda(x, y) - \frac{\delta_x(y)}{G^\lambda(x, \mathbf{1})} \right).$$

2.

$$P_{K^\lambda(x,\cdot)}(\tau_\Delta > n) = \left(1 - \frac{E_x[\exp(\lambda \tau_\Delta \wedge n) - 1]}{E_x[\exp(\lambda \tau_\Delta) - 1]} \right) e^{-\lambda n}, \ n \in \mathbb{Z}_+. \quad (5.1)$$

Proof. The first assertion is obtained by conditioning on the first step. As for the second,

$$\begin{aligned}
G^\lambda(x,\mathbf{1})P_{K^\lambda(x,\cdot)}(\tau_\Delta > n) &= \sum_{y\in S}\sum_{s=0}^{\infty} E_x[\mathbf{1}_{\{s<\tau_\Delta\}}\exp(\lambda s)\delta_y(X_s)]P_y(\tau_\Delta > n) \\
&= \exp(-\lambda n)E_x[\sum_{s=0}^{\infty}\exp(\lambda(s+n))\mathbf{1}_{\{\tau_\Delta > s+n\}}] \\
&= \exp(-\lambda n)\left(\sum_{s=n}^{\infty}\exp(\lambda s)E_x[\mathbf{1}_{\{\tau_\Delta > s\}}]\right) \\
&= \exp(-\lambda n)\left(G^\lambda(x,\mathbf{1}) - E_x[\sum_{s=0}^{(\tau_\Delta-1)\wedge(n-1)}\exp(\lambda s)]\right) \\
&= \exp(-\lambda n)\left(G^\lambda(x,\mathbf{1}) - \frac{E_x[\exp(\lambda\tau_\Delta\wedge n)-1]}{e^\lambda - 1}\right)
\end{aligned}$$

The result now follows because $G^\lambda(x,\mathbf{1}) = E_x[\exp(\lambda\tau_\Delta)-1]/(e^\lambda-1)$. □

Corollary 5.1. *Let* $\mathbf{x} = (x_n : n \in \mathbb{N})$ *satisfy* $\lim_{n\to\infty} x_n = \infty$. *Then the distribution of* τ_Δ *under* $P_{K^\lambda(x_n,\cdot)}$ *converges to Geom*$(1-e^{-\lambda})$ *if and only if* $\lim_{n\to\infty} E_{x_n}[\exp(\lambda\tau_\Delta)] = \infty$.

Proof. Suppose $\lim_{n\to\infty} E_{x_n}[\exp(\lambda\tau_\Delta)] = \infty$, then from (5.1), $P_{K^\lambda(x_n,\cdot)}(\tau_\Delta > t) \to e^{-\lambda t}$ for every $t \in \mathbb{Z}_+$. Otherwise, by switching to a subsequence, we may assume that $\lim_{n\to\infty} E_{x_n}[\exp(\lambda\tau_\Delta)]$ exists and is finite. Denote this limit by c, and note that $c \geq e^\lambda > 1$. Since for every $t \in \mathbb{N}$, $E_{x_n}[\exp(\lambda\tau_\Delta\wedge t)]-1 \geq e^\lambda - 1$, it follows that

$$\liminf_{n\to\infty}\frac{E_{x_n}[\exp(\lambda\tau_\Delta\wedge t)]-1}{E_{x_n}[\exp(\lambda\tau_\Delta)]-1} \geq \frac{e^\lambda-1}{c} > 0,$$

and so by (5.1),

$$\limsup_{n\to\infty} P_{K^\lambda(x_n,\cdot)}(\tau_\Delta > t) \leq (1-\frac{e^\lambda-1}{c})e^{-\lambda t}.$$

□

Proposition 5.2. *1. Let* $\mathbf{x} = (x_n : n \in \mathbb{N})$ *be a* λ,∞*-convergent sequence. Then* $\lim_{n\to\infty} K^\lambda(x_n,\cdot)$ *is a QSD if and only if* $(K^\lambda(x_n,\cdot) : n \in \mathbb{N})$ *is tight.*

2. Moreover, under the equivalent condition of part (1), $\lim_{n\to\infty} G^\lambda(x_n,\mathbf{1}) = \infty$.

Proof. We begin with the first assertion. Proposition 5.1, $\lim_{n\to\infty} x_n = \infty$ and Fatou's lemma give

$$(K^\lambda(\mathbf{x},\cdot)p)(y) \leq e^{-\lambda}K^\lambda(\mathbf{x},y).$$

Clearly, if $K^\lambda(\mathbf{x},\cdot)$ is a QSD, then $K^\lambda(\mathbf{x},\mathbf{1}) = 1$ which implies tightness. Conversely, tightness implies $K^\lambda(\mathbf{x},\mathbf{1}) = 1$, and the reverse inequality holds due to the argument given in the proof of Corollary 2.1, thus $K^\lambda(\mathbf{x},\cdot)$ is a QSD.

It remains to show that tightness is equivalent to the reverse inequality. If the sequence is tight, then we obtain the reverse inequality repeating the argument in the proof of Corollary 2.1 with ν_n replaced by $K^\lambda(x_n,\cdot)$. We omit details. Conversely, if the reverse inequality holds, we turn to the second assertion. Use Proposition 5.1 and the tightness, we obtain

$$\lim_{n\to\infty} P_{K^\lambda(x_n,\cdot)}(\tau_\Delta > t) = P_{K^\lambda(\mathbf{x},\cdot)}(\tau_\Delta > t),\ t \in \mathbb{Z}_+.$$

However, as $K^\lambda(\mathbf{x},\cdot)$ is a QSD with absorption parameter λ, the righthand side is $e^{-\lambda t}$. The conclusion now follows from Corollary 5.1. □

We are ready to prove the theorem.

5.2 Proof of Theorem 2.4 and Corollary 2.1

Proof of Theorem 2.4. We prove the two assertions in the order of appearance. Without loss of generality any sequence $\mathbf{X} = (x_n : n \in \mathbb{N})$ satisfying $\lim_{n\to\infty} x_n = \infty$ has a λ,∞-convergent subsequence. Pick any such subsequence, abusing notation, and denote it by $\mathbf{x}$. Next,

1. This key argument appeared in [10]. By assumption and Proposition 5.1,

$$\begin{aligned} E_{K^\lambda(x_n,\cdot)}[\exp(\lambda'\tau_\Delta)] &\le \sum_{t=0}^{\infty} e^{\lambda' t} P_{K^\lambda(x_n,\cdot)}(\tau_\Delta > t) \\ &\le \sum_{t=0}^{\infty} e^{(\lambda'-\lambda)t} \\ &= \frac{1}{1-e^{\lambda'-\lambda}}. \end{aligned}$$

   We denote the quantity on the righthand side by c. Now pick $\epsilon > 0$ and a finite $K = K(\epsilon)$, such that $E_x[\exp(\lambda\tau_\Delta)] > c/\epsilon$ for all $x \in K^c$. The lefthand side is clearly bounded below by $K^\lambda(x_n, K^c)c/\epsilon$, and therefore $K^\lambda(x_n, K^c) \le \epsilon$. Therefore the sequence $(K^\lambda(x_n,\cdot) : n \in \mathbb{N})$ is tight. Hence, Proposition 5.2 guarantees that $K^\lambda(\mathbf{x},\cdot)$ is a QSD.

2. We will argue by contradiction utilizing the reverse process introduced in Section 3.2. Assume that ν is a QSD with absorption parameter $\lambda > 0$. Rewriting 3.3-3 we obtain

$$\nu(x)E_x[\exp(\lambda\tau_\Delta)] = I_\nu(x).$$

   By assumption, the lefthand side is summable, and so is the righthand side. Using the definition of $I_\nu(x)$ (3.8) we then have

$$\sum_x I_\nu(x) = \sum_{z\in S} \nu(z)p(z,\Delta)(\sum_x Q_z({}^0\tau_x < \infty)).$$

The inner summation is the expectation of the number of states the path of the reverse process visits, which is always infinite (regardless of whether q is recurrent or - as in this case - transient), and since ν is strictly positive there exists at least one $z \in S$ with $p(z, \Delta) > 0$, the contradiction follows.

□

Proof of Corollary 2.1. From Theorem 2.4, there exists a QSD with survival parameter λ for all $\lambda \in (\lambda_0, \lambda_{cr})$. What remains to be shown is then the existence of a minimal QSD. To do that, suppose $\lambda_0 < \lambda_n \searrow \lambda_{cr}$ and let ν_n be a QSD with absorption parameter λ_n, whose existence is guaranteed from the theorem. Without loss of generality, we may assume that the sequence $(\nu_n(\cdot) : n \in \mathbb{N})$ converges pointwise to some limit. Denote this limit by ν. Fatou's lemma gives

$$\nu p \leq e^{-\lambda_{cr}} \nu.$$

To prove the reverse inequality, it is sufficient to show that $(\nu_n : n \in \mathbb{N})$ is tight. Indeed, if the sequence is tight, then for any $\epsilon > 0$, there exists a finite set K with $\nu_n(K) > 1 - \epsilon$ for all n and so

$$e^{-\lambda_n} \nu_n(y) = (\nu_n p)(y) = \sum_{x \in K} \nu_n(x) p(x, y) + \sum_{x \in K^c} \nu_n(x) p(x, y).$$

Denote the second sum on the righthand side by $H(y)$ and observe that $H(y) \geq 0$ and sums up to a number less than or equal to ϵ because p is substochastic. From bounded convergence we then have

$$e^{-\lambda_{cr}} \nu \leq \nu p + \epsilon,$$

and as ϵ is arbitrary the result follows. The key to proving the tightness rests on the fact that $\tau_\Delta \sim \text{Geom}(1 - e^{-\lambda_n})$ under P_{ν_n}. As the probability of success is increasing in n, for any fixed $\lambda \in (\lambda_0, \lambda_n)$, the sequence of MGFs, evaluated at λ, $E_{\nu_n}[\exp(\lambda \tau_\Delta)]$, is decreasing. Thus,

$$E_{\nu_1}[\exp(\lambda \tau_\Delta)] \geq E_{\nu_n}[\exp(\lambda \tau_\Delta)] \geq \nu_n(K^c) \inf_{x \in K^c} E_x[\exp(\lambda \tau_\Delta)]. \tag{5.2}$$

Denote the lefthand side of (5.2) by c. Pick $\epsilon > 0$. Now use the definition of λ_0 to pick a finite $K = K(\epsilon)$ so that $\inf_{x \in K^c} E_x[\exp(\lambda \tau_\Delta)] > c/\epsilon$. This gives $\nu_n(K^c) \leq \epsilon$, establishing tightness. □

5.3 Proof of Proposition 2.3 and Corollary 2.2

We begin by observing the following simple statements regarding $C^\lambda(x, y)$ defined in Definition 2.3.

First, by the FKG inequality [12]

$$E_x[\exp(\lambda \tau_\Delta), \tau_y < \tau_\Delta] \geq E_x[\exp(\lambda \tau_\Delta)] P_y(\tau_y < \tau_\Delta),$$

while on the other hand, $E_x[\exp(\lambda\tau_\Delta), \tau_y < \tau_\Delta] \le E_x[\exp(\lambda\tau_\Delta)]$. Therefore the coefficient $C^\lambda(x,y)$ satisfies

$$\frac{P_x(\tau_y < \tau_\Delta)}{1 - \frac{1}{E_x[\exp(\lambda\tau_\Delta)]}} \le C^\lambda(x,y) \le \frac{1}{1 - \frac{1}{E_x[\exp(\lambda\tau_\Delta)]}}. \tag{5.3}$$

Next,

$$\begin{aligned} K^\lambda(x,y) &= \frac{e^\lambda - 1}{E_x[\exp(\lambda\tau_\Delta)] - 1} \times \frac{E_x[\exp(\lambda\tau_y), \tau_y < \tau_\Delta]}{1 - E_y[\exp(\lambda\tau_y), \tau_y < \tau_\Delta]} \\ &= (e^\lambda - 1)\frac{C^\lambda(x,y)}{E_x[\exp(\lambda\tau_\Delta), \tau_y < \tau_\Delta]} \times \frac{E_x[\exp(\lambda\tau_y), \tau_y < \tau_\Delta]}{1 - E_y[\exp(\lambda\tau_y), \tau_y < \tau_\Delta]} \\ &= C^\lambda(x,y)\frac{e^\lambda - 1}{E_y[\exp(\lambda\tau_\Delta)](1 - E_y[\exp(\lambda\tau_y), \tau_y < \tau_\Delta])} \\ &= C^\lambda(x,y)\frac{e^\lambda - 1}{E_y[\exp(\lambda\tau_\Delta), \tau_\Delta < \tau_y]}, \end{aligned} \tag{5.4}$$

where the second line follows directly from the definition, and in the third we canceled the expectation of $e^{\lambda\tau_y}$ on the event $\tau_y < \tau_\Delta$ starting from x from both the numerator and the denominator.

Proof of Proposition 2.3. Recall the probability measure on S defined by (2.10), denoted by $K^\lambda(x,\cdot)$. Without loss of generality, we assume that $(K^\lambda(x_n,\cdot) : n \in \mathbb{N})$ converges pointwise to some limiting function $K^\lambda(\mathbf{x},\cdot)$. By either condition on S, $K^\lambda(\mathbf{x},\cdot)$ is not identically zero, and by Fatou's lemma $K^\lambda(\mathbf{x},\mathbf{1}) \le 1$. Next, for every $y \in S$, assumption (2.11) allows to apply the dominated convergence theorem to conclude that

$$K^\lambda(\mathbf{x},\cdot)p(y) = \lim_{n\to\infty} K^\lambda(x_n,\cdot)p(y) = e^{-\lambda}K^\lambda(\mathbf{x},y).$$

This proves the first assertion. Consider now the second assertion. Let $\nu = \frac{K^\lambda(\mathbf{x},\cdot)}{K^\lambda(\mathbf{x},\mathbf{1})}$. Then, ν is a QSD. Moreover, since by assumption and (5.4), $K^\lambda(\mathbf{x},y) \ge \frac{e^\lambda - 1}{E_y[\exp(\lambda\tau_\Delta), \tau_\Delta < \tau_y]}$ and $K^\lambda(\mathbf{x},\mathbf{1}) \le 1$, it follows that

$$\nu(y) \ge \frac{e^\lambda - 1}{E_y[\exp(\lambda\tau_\Delta), \tau_\Delta < \tau_y]}.$$

Yet Corollary 3.1 gives the reverse inequality. Hence, ν is given by the formula in the statement of the theorem. The uniqueness follows from the corollary too. Let $\tilde{\nu}$ be a QSD, then Corollary 3.1 gives $\tilde{\nu} \le \nu$, and since both are probability distributions, the equality follows. □

Proof of Corollary 2.2. Fix $0 < \lambda < \lambda_{cr}$. Clearly for every $x \notin A$, $E_x[\exp(\lambda\tau_\Delta), \tau_A < \tau_\Delta] = E_x[\exp(\lambda\tau_\Delta)]$. But $E_x[\exp(\lambda\tau_\Delta), \tau_A < \tau_\Delta] \le \sum_{y\in A} E_x[\exp(\lambda\tau_\Delta), \tau_y < \tau_\Delta]$, and therefore along any subsequence tending to infinity, there exists some y such that $C^\lambda(x,y) \ge \frac{1}{|A|}$ infinitely often, the result follows from Proposition 2.3-(1). □

Proof of Corollary 2.3. Let $(x_n : n \in \mathbb{N}) \in [\mathbf{x}]$. First assume $K^\lambda([\mathbf{x}], \cdot)$ is not identically zero. Pick $y \in S$ such that $K^\lambda([\mathbf{x}], y) > 0$. Thus, (5.4) gives $\lim_{n\to\infty} C^\lambda(x_n, y) \in (0, \infty)$.

For the converse, since by assumption $\lim_{n\to\infty} C^\lambda(x_n, y) > 0$, (5.4) guarantees that $K^\lambda([\mathbf{x}], y) > 0$. In addition, Proposition 5.1 gives

$$\sum_{z\in S} K^\lambda(x_n, z)p(z, y) = e^{-\lambda}\left(K^\lambda(x_n, y) - \frac{\delta_{x_n}(y)}{G^\lambda(x_n, \mathbf{1})}\right). \tag{5.5}$$

Since $K^\lambda(\cdot, \cdot)$ is nonnegative and bounded, the assumption (2.11) allows to invoke dominated convergence to conclude that $K^\lambda([\mathbf{x}], \cdot) = e^{-\lambda}K^\lambda([\mathbf{x}], \cdot)$. Therefore, we proved that $K^\lambda([\mathbf{x}], \cdot)$ is not identically zero, and satisfies (1.6). □

6 Examples

In this section, we provide several simple applications of our results.

6.1 Downward Skip-Free Chains

Consider a chain on $S = \mathbb{Z}_+$ and $\Delta = \{-1\}$, with the property that for every $x \in \mathbb{Z}_+$, and $l \in \{1, \ldots, x + 1\}$, we have $p(x, x - l) > 0$ if and only if $l = 1$. We will further assume that the chain satisfies Assumption **HD-1, HD-2**, and **HD-3**. Since by construction $\tau_\Delta \geq x$, Corollary 2.1 implies the existence of a QSD for every absorption parameter $\lambda \in (0, \lambda_{cr}]$. Moreover, Proposition 3.2-2 (with σ taken as the identity) and Corollary 3.1 guarantee that for each λ in this range, there exists a unique QSD with absorption parameter λ, ν_λ, given by the righthand side of (3.10). One notable case is of discrete time birth and death chains on $\mathbb{Z}_+ \cup \{-1\}$ absorbed at -1.

6.2 Generalized Cyclic Transfer

This is a concrete example of a skip-free chain and probably the simplest closed-form example. This process generalizes the cyclic transfer process from [15].

Assume $S = \mathbb{Z}_+$, let $q \in (0, 1)$ and μ be a probability distribution on S with unbounded support (if μ has finite support, all derivation in this section hold verbatim with $S = \{0, 1, \ldots, \max \mathrm{Supp}(\mu)\}$). For $x \in S$, consider the transition function p on $S \cup \{\Delta\}$ illustrated in Figure 1 and given by

$$\begin{cases} p(x, x-1) = 1 & x \in \mathbb{N} \\ p(0, \Delta) = q & \\ p(0, x) = (1-q)\mu(x) & x \in S \end{cases}$$

Let φ_μ be the moment generating function for μ:

$$\varphi_\mu(\lambda) = \sum_{j=0}^{\infty} \mu(j)e^{\lambda j}. \tag{6.1}$$

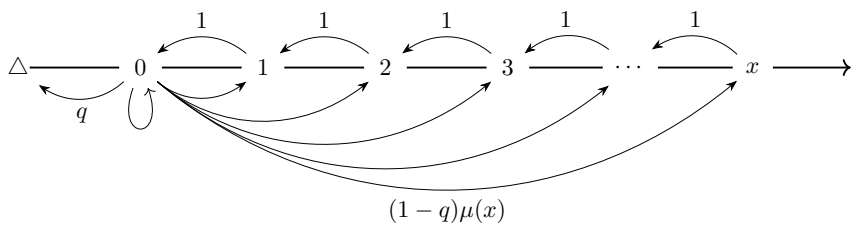

Figure 1: Cyclic Transfer

We will assume that $\varphi_\mu(\lambda) < \infty$ for some $\lambda > 0$. Thus, **HD-1, HD-2** and **HD-3** hold. Let $\lambda \in (0, \lambda_{cr})$. Observe that

$$E_0[\exp(\lambda\tau_\Delta)] = e^\lambda q + e^\lambda(1-q)\varphi_\mu(\lambda)E_0[\exp(\lambda\tau_\Delta)].$$

Therefore

$$E_0[\exp(\lambda\tau_\Delta)] = \frac{e^\lambda q}{1 - e^\lambda(1-q)\varphi_\mu(\lambda)}. \tag{6.2}$$

Hence, λ_{cr} is the unique solution to

$$(1-q)e^\lambda\varphi_\mu(\lambda) = 1. \tag{6.3}$$

This implies that λ_{cr} is in the infinite MGF regime.

Next, since the process is downward-skip free, we will apply equation (3.10) to obtain the unique QSD given by

$$\nu_\lambda(y) = \frac{e^\lambda - 1}{E_y[\exp(\lambda\tau_\Delta), \tau_\Delta < \tau_y]},$$

with absorption parameter λ, for each $\lambda \in (0, \lambda_{cr}]$. To obtain an explicit formula, a similar calculation shows that for all $y \in \mathbb{Z}_+$

$$E_0[\exp(\lambda\tau_\Delta), \tau_\Delta < \tau_y] = e^\lambda q + e^\lambda \sum_{0 \le j < y} (1-q)\mu(j)e^{\lambda j}E_0[\exp(\lambda\tau_\Delta), \tau_\Delta < \tau_y],$$

and therefore

$$E_0[\exp(\lambda\tau_\Delta), \tau_\Delta < \tau_y] = \frac{e^\lambda q}{1 - e^\lambda \sum_{j=0}^{y-1}(1-q)\mu(j)e^{\lambda j}}.$$

In addition,

$$\begin{aligned} E_y[\exp(\lambda\tau_\Delta), \tau_\Delta < \tau_y] &= e^{\lambda y}E_0[\exp(\lambda\tau_\Delta), \tau_\Delta < \tau_y] \\ &= e^{\lambda y}(e^\lambda q + e^\lambda \sum_{j=0}^{y-1}(1-q)\mu(j)e^{\lambda j}E_0[\exp(\lambda\tau_\Delta), \tau_\Delta < \tau_y]) \end{aligned}$$

So

$$E_y[\exp(\lambda\tau_\Delta, \tau_\Delta < \tau_y] = \frac{e^{\lambda(y+1)}q}{1 - e^\lambda \sum_{j=0}^{y-1}(1-q)\mu(j)e^{\lambda j}}.$$

Thus,

$$\nu_\lambda(y) = \frac{1}{q}e^{-\lambda(y+1)}(e^\lambda - 1)(1 - (1-q)\sum_{0\le j<y}\mu(j)e^{\lambda(j+1)}). \tag{6.4}$$

In summary, λ_{cr} is given by (6.3), is in the infinite MGF regime and for each $\lambda \in (0, \lambda_{cr}]$ there exists a unique QSD given by (6.4). Note that the existence of all these QSDs is an immediate application of Corollary 2.1.

6.3 Absorption Probability Bounded from Below

Consider any process $\mathbf{Y}$ on $S \cup \{\Delta\}$ with transition function p^Y satisfying Assumption **HD-1, HD-2** and **HD-3**. We will also assume that for any $\lambda \in (0, \lambda_{cr}^Y)$, $\lim_{x\to\infty} E_x[\exp(\lambda\tau_\Delta^Y)] = \infty$. Note that we have used the superscript Y to denote quantities associated with Y, as we now introduce the process $\mathbf{X}$.

Let J be a geometric random variable, independent of $\mathbf{Y}$ with probability of success $1 - e^{-\rho}$ for some $\rho > 0$. Define $\mathbf{X}$ as follows:

$$X_n = \begin{cases} Y_n & n < J \\ \Delta & \text{otherwise} \end{cases}$$

This is equivalent to defining $p(x,y) = e^{-\rho}p^Y(x,y)$ for $x, y \in S$, and $p(x,\Delta) = 1 - \sum_{y\in S} p(x,y)$. Clearly,

$$P_x(\tau_\Delta > n) = P(\tau_\Delta^Y \wedge J > n) = P_x(\tau_\Delta > n)P(J > n) = P_x(\tau_\Delta > n)e^{-n\rho}.$$

Now since for every random variable T which is nonnegative and taking integer values we have

$$E[\exp(\lambda T)] = 1 + (e^\lambda - 1)\sum_{n=0}^{\infty} e^{\lambda n}P(T > n),$$

it follows that

$$\begin{aligned} E_x[\exp(\lambda\tau_\Delta)] &= 1 + (e^\lambda - 1)\sum_{n=0}^{\infty} e^{(\lambda-\rho)n}P_x(\tau_\Delta^Y > n) \\ &= 1 + \frac{e^\lambda - 1}{e^{\lambda-\rho} - 1}(E_x[\exp((\lambda-\rho)\tau_\Delta^Y)] - 1). \end{aligned}$$

Therefore this expression is bounded as a function of x if $\lambda < \rho$, and is equal to $1+(e^\lambda-1)E_x[\tau_\Delta^Y]$ when $\lambda = \rho$, and tends to infinity as $x \to \infty$ when $\lambda > \rho$. Also, $\lambda_{cr} = \lambda_{cr}^Y + \rho$. It follows from Corollary 2.1 that $\mathbf{X}$ has QSDs with absorption parameter λ for every $\lambda \in (\rho, \rho + \lambda_{cr}^Y]$, and that it does not possess any QSDs with absorption parameter $\lambda \in (0, \rho)$.

6.4 Subcritical Branching Process

Let $\mathbf{X}$ be a branching process with a nondegenerate offspring distribution B on $\mathbb{Z}_+$ (we will abuse notation and will refer to B as a random variable). As usual [2, p. 3] and to avoid trivialities we will assume $P(B = j) < 1$ for all $j \in \mathbb{N}$ and $P(B \leq 1) < 1$. The unique absorbing state is 0. We will also assume that the process is subcritical, namely $E[B] \in (0, 1)$, and let $m = E[B]$. A straightforward calculation shows that for any $x \in \mathbb{N}$, $E_x[e^{\lambda\tau_\Delta}] < \infty$ if $e^\lambda < \frac{1}{m}$ and $E_x[e^{\lambda\tau_\Delta}] = \infty$ if $e^\lambda > \frac{1}{m}$. Therefore $e^{\lambda_{cr}} = \frac{1}{m}$, equivalently $\lambda_{cr} = \ln \frac{1}{m}$. Though the restriction of $\mathbf{X}$ to the non-absorbing set $\mathbb{N}$ is not irreducible, we can restrict the process to an infinite subset of $\mathbb{N}$ depending on the support of B, resulting in an irreducible process.

Our results provide a quick way to prove the existence of a continuum of QSDs. Indeed, for any $\lambda \in (0, \lambda_{cr})$, Jensen's inequality gives $E_x[\exp(\lambda\tau_\Delta)] \geq e^{\lambda E_x[\tau_\Delta]}$. As $E_x[\tau_\Delta]$ is the expectation of the maximum of x independent copies of τ_Δ under the distribution P_1, it immediately follows that $\lim_{x\to\infty} E_x[\tau_\Delta] = \infty$, and therefore Corollary 2.1 holds with $\lambda_0 = 0$. Namely, for every $\lambda \in (0, \lambda_{cr} = \ln \frac{1}{m}]$ there exists a QSD with absorption parameter λ.

Existence and convergence results for a minimal QSD for the subcritical branching process are among the earliest in the field of QSDs. Let f be the generating function of B. Yaglom's theorem [2, Corollary 1, p. 18] states that for $x \in \mathbb{N}$, $P_x(X_n \in \cdot \mid \tau_\Delta > n)$ converges as $n \to \infty$ to a probability distribution on $\mathbb{N}$ which is the unique solution to the functional equation

$$\mathcal{B}(f(s)) = m\mathcal{B}(s) + (1 - m) \tag{6.5}$$

among all probability distributions on $\mathbb{N}$. Being obtained as a quasi-limiting distribution, this limit is also a QSD. A straightforward calculation of the generating function for a solution to (1.6) with $e^{-\lambda} = m$ reveals that it must satisfy (6.5), and so a minimal QSD exists and is unique. Denote this QSD by ν_{cr}.

As is well known, [2, Corollary 2, p. 45], the additional assumption $E[B \ln(1+B)] < \infty$ is equivalent to ν_{cr} having finite expectation, namely $\sum_{i=1}^\infty \nu_{cr}(i)i < \infty$. As the identity function $h(i) = i$ on $\mathbb{N}$ satisfies $ph = mh$, a straightforward application of the definition of the reverse chain associated with ν_{cr}, (3.5), reveals that under this additional condition, $\nu_{cr}h$ can be normalized to be a stationary distribution for q. Hence, q is positive recurrent and proves the existence of a minimal QSD. Proposition 3.3 and the comment below it guarantee that λ_{cr} is in the infinite regime and that (2.1) and (2.3) hold. Moreover, if we take S as the irreducible non-absorbing class mentioned above, then since the $P(B = 0) > 0$, it follows that for every state $x \in \mathbb{N}$ in the support of B, $p(x, x) > 0$, which along the irreducibility on S, implies that $\mathbf{X}$ is aperiodic. Thus, both the representation and the convergence results in Theorem 2.1 hold, and in particular, (2.2) provides us with a new MCMC method for sampling from the minimal QSD under this additional condition. We refer the reader to [13], which discusses numerical methods for solving (6.5).

6.5 Rooted Tree

Consider an infinite rooted tree, with the root ρ_r being the only state from which absorption is possible. We will assume p is the nearest neighbor Markov Chain on this tree with a unique absorption state Δ satisfying Assumption **HD-1, HD-2, HD-3** and condition 2.11. For example, we can assume the degree of each vertex is bounded, and the transition from any vertex on the tree to any neighboring vertex on the tree is strictly positive, and for vertices other than the root, the transition to the unique vertex closer to the root is uniformly bounded below by $\frac{1}{2} + \epsilon$ for some $\epsilon > 0$.

Suppose λ is in the finite regime (which may include λ_{cr}), and take a sequence of vertices $(x_n : n \in \mathbb{N})$ going to infinity along some unique branch. Recall $C^\lambda(x, y)$ from Definition 2.3. Two alternatives are illustrated by the following specific graph:

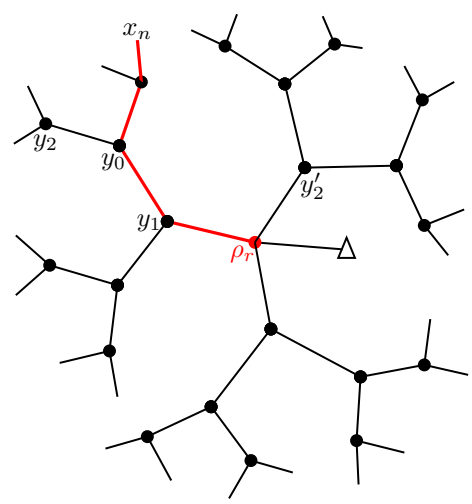

Figure 2: Regular tree of degree 3 with a unique absorption state Δ

1. For y in that branch, we clearly have $\lim_{n\to\infty} C^\lambda(x_n, y) = 1$. For instance, y_1 is on the branch in Figure 2.

2. For other y, we need to consider two cases.

   (a) y is not on the branch, yet it has ancestors on the branch other than ρ_r. One example is the vertex y_2 in Figure 2. In this case, let y_0 denote the most recent ancestor of y_2 on the branch. For a path to get to y_2, it must pass through y_0. With this,

$$C^\lambda(x_n, y) = \frac{E_{x_n}[\exp(\lambda\tau_{y_0}), \tau_{y_0} < \tau_\Delta] E_{y_0}[\exp(\lambda\tau_y), \tau_y < \tau_\Delta] E_y[\exp(\lambda\tau_\Delta)]}{E_{x_n}[\exp(\lambda\tau_\Delta)] - 1}$$

$$\underset{n\to\infty}{\to} \frac{E_{y_0}[\exp(\lambda\tau_y), \tau_y < \tau_\Delta] E_y[\exp(\lambda\tau_\Delta)]}{E_{y_0}[\exp(\lambda\tau_\Delta)]} < 1$$

(b) y has no ancestor on the branch other than ρ_r, and so to get to y from x_n with n large enough, we need to pass through the root where absorption is possible. For instance, in Figure 2 y_2' has no ancestor on the branch, and in order to go from x_n to y_2', we need to pass through ρ_r. Thus,

$$C^\lambda(x_n, y) = \frac{E_{x_n}[\exp(\lambda\tau_{\rho_r})]E_{\rho_r}[\exp(\lambda\tau_\Delta), \tau_y < \tau_\Delta]}{E_{x_n}[\exp(\lambda\tau_{\rho_r})]E_{\rho_r}[\exp(\lambda\tau_\Delta)] - 1}$$
$$\underset{n\to\infty}{\to} \frac{E_{\rho_r}[\exp(\lambda\tau_\Delta), \tau_y < \tau_\Delta]}{E_{\rho_r}[\exp(\lambda\tau_\Delta)]} < 1.$$

Therefore, by Proposition 2.3, for each branch, we have a QSD corresponding to that branch.

Next, recall $K^\lambda(x_n, \cdot)$ from Definition 2.1, we will also show that in this finite MGF regime, for every y, $\lim_{n\to\infty} K^\lambda(x_n, y)$ exists and is a QSD.

Let $\bar{y}$ be the unique element on the same branch as $(x_n : n \in \mathbb{N})$ satisfying $|\bar{y}| = |y|$, i.e., the lengths of the shortest paths from ρ_r are equal. Since we are interested in the limit as $n \to \infty$, without loss of generality, we can assume $|x_n| > |y|$ for all n. To reach y from x_n, the process must first hit $\bar{y}$. Therefore,

$$G^\lambda(x_n, y) = E_{x_n}[\exp(\lambda\tau_{\bar{y}})] \times \frac{E_{\bar{y}}[\exp(\lambda^0\tau_y),^0 \tau_y < \tau_\Delta]}{1 - E_y[\exp(\lambda\tau_y), \tau_y < \tau_\Delta]}.$$

Also,

$$G^\lambda(x_n, \mathbf{1}) = E_{x_n}[\exp(\lambda\tau_{\bar{y}})]G^\lambda(\bar{y}, \mathbf{1}).$$

This gives

$$K^\lambda(x_n, y) = \frac{G^\lambda(x_n, y)}{G^\lambda(x_n, \mathbf{1})} = \frac{E_{\bar{y}}[\exp(\lambda^0\tau_y),^0 \tau_y < \tau_\Delta]}{G^\lambda(\bar{y}, \mathbf{1})} \times \frac{1}{1 - E_y[\exp(\lambda\tau_y), \tau_y < \tau_\Delta]}. \tag{6.6}$$

As the limit trivially exists, by Corollary 2.3, it is a QSD. Therefore, there exist QSDs with absorption parameter λ, and all such QSDs can be obtained through Theorem 2.3, where the elements of S^λ can be indexed by the branches, and are each given by the righthand side of (6.6), where y is any vertex and $\bar{y}$ is the unique vertex on the branch satisfying $|\bar{y}| = |y|$.

7 Results: Continuous-Time

7.1 Definitions and Assumptions

We adapt the main results of Section 2 to the continuous-time setting. This adaptation is mostly straightforward and routine, and we present it primarily in order to make a connection with the large body of literature in the continuous-time setting.

We begin by introducing the set of hypotheses. Let $\mathbb{X} = (\mathbb{X}_t : t \in \mathbb{R}_+)$ be a Markov Chain on a state space which is a disjoint union $S \cup \{\Delta\}$, where S is either finite or countably infinite. We will denote the distribution and expectation of $\mathbb{X}$ under the initial distribution μ by P_μ and E_μ respectively, with P_x and E_x serving as shorthand for P_{δ_x} and E_{δ_x}, respectively. For $x \in S \cup \{\Delta\}$, let

$$\mathbb{T}_x = \inf\{t > 0 : \mathbb{X}_t = x, \mathbb{X}_{t-} \neq x\}. \tag{7.1}$$

We will work under the following hypotheses, which are the analogs of the assumptions made for the discrete-time setting:

HC-1. *$\mathbb{T}_\Delta < \infty$, $P_x - a.s.$ for some $x \in S$.*

HC-2. *The set S is an irreducible class, and the exponential holding time at each $x \in S$ has parameter $q_x \in (0, \infty)$.*

HC-3. *There exists $\beta > 0$ such that $E_x[\exp(\beta \mathbb{T}_\Delta)] < \infty$ for some (equivalently all) $x \in S$.*

Clearly, Δ is the unique absorbing state, and therefore, we will refer to $\mathbb{T}_\Delta$ as the absorption time. Note that **HC-3** guarantees that $\mathbb{X}$ is non-explosive. This is mostly for the simplicity of the presentation, as the explosion can be handled by the discretization scheme we use to derive the results below. We briefly review the notion of a QSD in a continuous time setting and some basic properties. Recall that a probability distribution ν on S is a QSD if the following analog of (1.3) holds.

$$P_\nu(\mathbb{X}_t \in \cdot \mid \mathbb{T}_\Delta > t) = \nu(\cdot), \ t \in \mathbb{R}_+, \tag{7.2}$$

If ν is a QSD, then under P_ν, $\mathbb{T}_\Delta$ is exponentially distributed with parameter $\lambda > 0$. That is, the following analog of (1.5) holds:

$$P_\nu(\mathbb{T}_\Delta > t) = e^{-\lambda t}, \ t \in \mathbb{R}_+, \tag{7.3}$$

and we say that ν is a QSD with absorption parameter λ.

We write $(\mathcal{P}_t : t \in \mathbb{R}_+)$ for the semigroup of contractions on $\ell^1(S)$ given by

$$(\nu \mathcal{P}_t)(y) = \sum_{x \in S} \nu(x) P_x(\mathbb{X}_t = y), \ \nu \in \ell^1(S).$$

Assumption $\mathbf{HC-2}$ implies that the semigroup is weakly continuous from the right at 0. Namely, for any $\nu \in \ell^1(S)$ and $f \in \ell^\infty(S)$, $\lim_{t \downarrow 0} (\nu \mathcal{P}_t) f = \lim_{t \downarrow 0} \sum_{x \in S} \nu(x) E_x[f(\mathbb{X}_t)] = \sum_{x \in S} \mu(x) f(x)$, and therefore, [21, p. 255, Theorem], [14, Chapter X, Corollary of Theorem 10.2.3], the semigroups is strongly continuous, that is for any $\nu \in \ell^1(S)$, $\lim_{t \downarrow 0} \nu \mathcal{P}_t = \nu$ in $\ell^1(S)$. As a result, the semigroup has a densely defined generator $\mathcal{L}$:

$$\nu \mathcal{L} = \lim_{t \downarrow 0} \frac{\nu \mathcal{P}_t - \nu}{t}, \tag{7.4}$$

where the limit is in $\ell^1(S)$. In particular, we have the following

Proposition 7.1. *A probability measure ν on S is a QSD for $\mathbb{X}$ with absorption parameter $\lambda > 0$ if and only if ν is in the domain of $\mathcal{L}$ and*

$$\nu\mathcal{L} = -\lambda\nu. \tag{7.5}$$

Though the proof is straightforward, we bring it here for completeness. Clearly, (7.5) can be interpreted as a sequence of linear equations ([26, p. 691-692] provides an example of a birth and death process and a probability measure satisfying the sequence of equations but which is not a QSD. In the present context, the latter statement means that the probability measure is not in the domain of $\mathcal{L}$).

Proof. Assume first that ν is a QSD. Then (7.2) can be rewritten as

$$\nu\mathcal{P}_t = \sum_y (\nu\mathcal{P}_t)(y)\nu.$$

Thus, on the one hand, $\nu\mathcal{P}_{t+s} = \sum_y (\nu\mathcal{P}_{t+s})(y)\nu$, while on the other hand, using the semigroup property,

$$\begin{aligned}\nu\mathcal{P}_{t+s} &= \nu\mathcal{P}_t\mathcal{P}_s \\ &= \sum_y (\nu\mathcal{P}_t)(y)\nu\mathcal{P}_s \\ &= \sum_y (\nu\mathcal{P}_t)(y) \sum_y (\nu\mathcal{P}_s)(y)\nu.\end{aligned}$$

As ν is strictly positive, it follows that the function $t \to \sum_y (\nu\mathcal{P}_t)(y)$ is multiplicative. It is equal to 1 at zero and nonincreasing and tends to 0 as $t \to \infty$. It is therefore of the form $e^{-\lambda t}$ for some $\lambda > 0$. As a result, $\nu\mathcal{P}_t = e^{-\lambda t}\nu$. This implies that ν is in the domain of $\mathcal{L}$ and that (7.5) holds. Conversely, if ν is in the domain of $\mathcal{L}$ and satisfies (7.5), then [21, Section 2, Theorem 1.3] gives that for any $f \in \ell^\infty(S)$, $\nu\mathcal{P}_t f = \nu f + \int_0^t \nu\mathcal{L}\mathcal{P}_s f ds$, that is the continuous function $\phi_f(t) = \nu\mathcal{P}_t f$ on $\mathbb{R}_+$ satisfies $\phi_f(t) = \phi_f(0) - \lambda \int_0^t \phi_f(s)ds$, and so $\phi_f(t) = e^{-\lambda t}\phi_f(0)$. Equivalently, $\nu\mathcal{P}_t = e^{-\lambda t}\nu$, which implies (7.2). □

Next we define the critical absorption parameter λ_{cr} analogously to (1.7):

$$\lambda_{cr} = \sup\{\lambda > 0 : E_x[\exp(\lambda\tau_\Delta)] < \infty \text{ for some } x \in S\}. \tag{7.6}$$

Note that **HC-3** implies that the exponential parameter of the holding time at each state x, q_x, is strictly larger than β, and as a result, $\inf_{x\in S} q_x \geq \beta$, so $\lambda_{cr} \in (0, \inf_{x\in S} q_x]$. As before, we say that $\lambda > 0$ is in the finite MGF regime if $E_x[\exp(\lambda\tau_\Delta)] < \infty$ for some (equivalently all) $x \in S$, and that λ_{cr} is in the infinite MGF regime if for some (equivalently all) $x \in S$, $E_x[\exp(\lambda_{cr}\tau_\Delta)] = \infty$. A QSD with absorption parameter λ_{cr} is called the minimal QSD.

We record the following analog of Proposition 1.2, which we will need in the sequel. As the proof is identical to the proof in the discrete case, it will be omitted.

Proposition 7.2. *Let $\lambda \in (0, \lambda_{cr}]$. Then for every $x \in S$, $E_x[\exp(\lambda\tau_x), \mathbb{T}_x < \mathbb{T}_\Delta] \leq 1$. Moreover,*

1. *If $E_x[\exp(\lambda\mathbb{T}_\Delta)] < \infty$ then the inequality is strict;*
2. *If $E_x[\exp(\lambda_{cr}\tau_\Delta)] = \infty$ and $E_x[\exp(\lambda_{cr}\mathbb{T}_\Delta), \mathbb{T}_\Delta < \mathbb{T}_x] < \infty$ for some $x \in S$, then $E_x[\exp(\lambda_{cr}\mathbb{T}_x), \mathbb{T}_x < \tau_\Delta] = 1$ for all $x \in S$.*

7.2 Discretizing Time

All results we present below use the embedded discrete-time processes we present in this section. For $d > 0$, define a discrete-time Markov chain on $S \cup \{\Delta\}$, $\mathbf{X}^d = (X_n^d : n \in \mathbb{Z}_+)$ by letting

$$X_n^d = \mathbb{X}_{dn}. \tag{7.7}$$

For $x \in S \cup \{\Delta\}$, let

$$\tau_x^d = \inf\{n \in \mathbb{N} : X_n^d = x\}.$$

Clearly, $\mathbf{X}^d$ is a Markov chain satisfying **HD-1, HD-2** and **HD-3**. Moreover, if $\tau_\Delta^d = n$, then on the one hand necessarily, $\mathbb{X}_{nd} = \Delta$, and so $\mathbb{T}_\Delta \leq dn$, and on the other hand, $X_{n-1}^d \in S$ or, equivalently, $\mathbb{X}_{(n-1)d} \in S$, so $\mathbb{T}_\Delta > d(n-1)$. We therefore have that for every $x \in S$ and $d > 0$,

$$\mathbb{T}_\Delta \leq d\tau_\Delta^d < \mathbb{T}_\Delta + d. \tag{7.8}$$

From this, it follows that

$$E_x[\exp(\lambda\mathbb{T}_\Delta)] \leq E_x[\exp(\lambda d\tau_\Delta^d)] \leq e^{\lambda d} E_x[\exp(\lambda\mathbb{T}_\Delta)]. \tag{7.9}$$

and in particular letting λ_{cr}^d denote the critical absorption parameter for $\mathbf{X}^d$, then

$$\lambda_{cr}^d = d\lambda_{cr}. \tag{7.10}$$

The following allows connecting hitting times of states other than Δ for the continuous and the discrete processes.

Proposition 7.3. *Let $x_0 \in S$ and suppose that $E_{x_0}[\exp(\lambda\mathbb{T}_\Delta \wedge \mathbb{T}_{x_0})] < \infty$. Then there exists $d_0 > 0$ such that for $d \in (0, d_0)$, $E_{x_0}[\exp(\lambda d(\tau_\Delta^d \wedge \tau_{x_0}^d))] < \infty$.*

Proof. We begin by recalling that if $\mathbb{X}$ starts from x_0, then the state where it jumps to first and the time it takes to perform the jump are independent. This can be done through the construction of the Markov chain. For simplicity, label the states other than x_0 by $1, 2, \ldots$. Let $T_1, T_2, \ldots$ be independent exponential random variables with respective parameters $\rho_1, \rho_2, \ldots$. We assume $q = q_{x_0} = \sum_j \rho_j < \infty$. Let $T = \inf T_j$. Clearly, T is exponential with parameter q. Also, let R be the smallest (and unique, almost surely) index j satisfying $T = T_j$. Then, the chain will jump to state R at time T. The joint distribution of R and T is given by

$$P(R = j, T > t) = P(T_j > t, \cap_{i \neq j}\{T_i \geq T_j\}) = \rho_j \int_t^\infty e^{-\rho_j s} e^{-\sum_{i \neq j} \rho_i s} ds = \frac{\rho_j}{q} e^{-qt}.$$

Therefore, R and T are independent. In particular, $P(R = j|T \leq d) = P(R = j) = \frac{\rho_j}{q}$.

We turn to the main claim. Let $\mathbb{T}^1_{x_0} = \mathbb{T}_{x_0}$ and continue inductively, defining a sequence $(\mathbb{T}^k_{x_0} : k = 1, 2, \dots)$ by letting

$$\mathbb{T}^{k+1}_{x_0} = \inf\{t > \mathbb{T}^k_{x_0} : \mathbb{X}_{t-} \neq x_0,\ \mathbb{X}_t = x_0\}.$$

Let $J = \inf\{k \geq 1 : \mathbb{T}^k_{x_0} < \infty, \mathbb{X}_{\mathbb{T}^k_{x_0}+s} = x_0,\ \text{for all } s \in [0, d]\}$. In both definitions above, we adopt the convention $\inf \emptyset = \infty$. Note that J is the first visit to x_0 after which the $\mathbb{X}$ stays put for at least d units, and recall that the distribution of a Markov Chain right after a jump from a given state is independent of the time it took to jump from the state. On the event $J < \infty$, $\mathbb{T}^J_{x_0}$ is finite, and on $J = \infty$, we define $\mathbb{T}^J_{x_0} = \infty$. We need a few more definitions. First, for $M > 0$, let $v_M(x_0) = E_{x_0}[\exp(\lambda(\mathbb{T}^J_{x_0} \wedge \mathbb{T}_\Delta \wedge M))]$.

Let $S = E_{x_0}[\exp(\lambda(\mathbb{T}_{x_0} \wedge \mathbb{T}_\Delta)), \mathbb{T}_\Delta < \mathbb{T}_{x_0}] + E_{x_0}[\exp(\lambda \mathbb{T}_{x_0} \wedge \mathbb{T}_\Delta)), \mathbb{T}_{x_0} < \mathbb{T}_\Delta, J = 1]$. This expression is bounded above by $E_{x_0}[\exp(\lambda(\mathbb{T}_{x_0} \wedge \mathbb{T}_\Delta))]$ and is therefore finite by assumption. We have the upper bound

$$v_M(x_0) \leq S + E_{x_0}[\exp(\lambda(\mathbb{T}^J_{x_0} \wedge \mathbb{T}_\Delta)), \mathbb{T}_{x_0} < \mathbb{T}_\Delta, J > 1]. \tag{7.11}$$

We examine the second summand on the righthand side. Letting

$$\eta_d = E_{x_0}[\exp(\lambda(\mathbb{T}_{x_0} \wedge \mathbb{T}_\Delta)), \mathbb{T}_{x_0} < \mathbb{T}_\Delta](1 - e^{-q_{x_0} d})e^{\lambda d},$$

then the second summand on the righthand side of (7.11) is bounded above by $\eta_d v_M(x_0)$. In addition, there exists d_0 such that for $d < d_0$, $\eta_d < 1$. From these bounds we conclude that $v_M(x_0) \leq S + \eta_d v_M(x_0)$, and since $v_M(x_0)$ is finite by construction, $v_M(x_0) \leq \frac{S}{1-\eta_d}$. The righthand side is independent of M. By letting $M \to \infty$, it follows from monotone convergence that

$$E_{x_0}[\exp(\lambda(\mathbb{T}^J_{x_0} \wedge \tau_\Delta))] \leq \frac{S}{1 - \eta_d} < \infty.$$

As it is always true that $d\tau^d_\Delta \leq \mathbb{T}_\Delta + d$, on the event $J = \infty$, we clearly have $d(\tau^d_{x_0} \wedge \tau^d_\Delta) \leq \mathbb{T}^J_{x_0} \wedge \mathbb{T}_\Delta + d$. On the event $J < \infty$, the definition of $\mathbb{T}^J_{x_0}$ gives $d\tau^d_{x_0} \leq \mathbb{T}^J_{x_0} + d$, so $d(\tau^d_{x_0} \wedge \tau^d_\Delta) \leq \mathbb{T}^J_{x_0} \wedge \mathbb{T}_\Delta + d$ too. This completes the proof. □

Finally, we provide a connection between the QSDs for $\mathbb{X}$ and those for $\mathbf{X}^d$.

Proposition 7.4. *Let $d > 0$. The process $\mathbb{X}$ has a QSD with absorption parameter λ if and only if $\mathbf{X}^d$ has a QSD with absorption parameter $d\lambda$. Specifically,*

1. *Let ν be a QSD for $\mathbb{X}$ with absorption parameter λ. Then ν is a QSD for $\mathbf{X}^d$ with absorption parameter $d\lambda$.*
2. *Conversely, let ν be a QSD for $\mathbf{X}^d$ with absorption parameter $d\lambda$. Then $\int_0^d e^{\lambda s} P_\nu(\mathbb{X}_s \in \cdot\ )ds$ can be normalized to be a QSD for $\mathbb{X}$ with absorption parameter λ.*

Proof. The first numbered assertion is trivial. As for the second, suppose that ν is a QSD for $\mathbf{X}^d$ with absorption parameter λd. For $s \in \mathbb{R}_+$ and $x \in S$, let $h(s,x) = e^{\lambda s} P_\nu(\mathbb{X}_s = x)$. Then

$$\begin{aligned} h(s+d,x) &= e^{\lambda s} e^{\lambda d} P_\nu(\mathbb{X}_{s+d} = x) \\ &= e^{\lambda s} e^{\lambda d} E_\nu P_{X_1^d}(\mathbb{X}_s = x) \\ &= e^{\lambda s} P_\nu(\mathbb{X}_s = x) = h(s,x). \end{aligned}$$

Note here that we pass from the second line to the third using the fact that ν is a QSD for $\mathbf{X}^d$, and so for any bounded f on $S \cup \{\Delta\}$ vanishing on Δ we have $e^{\lambda d} E_\nu[f(X_1^d)] = E_\nu[f(X_0^d)]$. Here we used $f(u) = P_u(\mathbb{X}_s = x)$. We proved that h is d-periodic in the first variable. Define $\tilde{\nu}(x) = \int_0^d h(s,x)ds$. Then $\tilde{\nu}$ is a finite measure on S. Also, for $t \in \mathbb{R}_+$ and $y \in S$

$$\begin{aligned} \sum_x \tilde{\nu}(x) P_x(\mathbb{X}_t = y) &= \sum_x \int_0^d e^{\lambda s} P_\nu(X_s = x) P_x(X_t = y) ds \\ &= e^{-\lambda t} \int_0^d h(s+t,y) ds \\ &\overset{u=s+t}{=} e^{-\lambda t} \int_t^{t+d} h(u,y) du \\ &= e^{-\lambda t} \tilde{\nu}(y). \end{aligned}$$

Thus, by normalizing $\tilde{\nu}$, we obtain a QSD with absorption parameter λ. □

7.3 Infinite MGF Regime

We begin with the analog of Theorem 2.1

Theorem 7.1. *Suppose $\mathbb{\lambda}_{cr}$ is in the infinite MGF regime. Then*

1. *There exists a minimal QSD if and only if there exists $x \in S$ such that*

$$E_x[\exp(\mathbb{\lambda}_{cr} \mathbb{\tau}_\Delta \wedge \mathbb{\tau}_x)] < \infty. \tag{7.12}$$

In this case, there exists a unique minimal QSD $\mathbb{\nu}_{cr}$, which is given by the formula

$$\begin{aligned} \mathbb{\nu}_{cr}(x) &= \frac{\mathbb{\lambda}_{cr}}{q_x - \mathbb{\lambda}_{cr}} \frac{1}{E_x[\exp(\mathbb{\lambda}_{cr} \mathbb{\tau}_\Delta \wedge \mathbb{\tau}_x)] - 1} \\ &= \frac{\mathbb{\lambda}_{cr}}{q_x - \mathbb{\lambda}_{cr}} \frac{1}{E_x[\exp(\mathbb{\lambda}_{cr} \mathbb{\tau}_\Delta), \mathbb{\tau}_\Delta < \mathbb{\tau}_x]}. \end{aligned} \tag{7.13}$$

2. *If, in addition to* (7.12),

$$E_x[\exp(\mathbb{\lambda}_{cr} \mathbb{\tau}_x) \mathbb{\tau}_x, \mathbb{\tau}_x < \mathbb{\tau}_\Delta] < \infty \text{ for some } x \in S, \tag{7.14}$$

then for any finitely supported μ on S,

$$\lim_{t\to\infty} P_\mu(\mathbb{X}_t \in \cdot \mid \mathbb{T}_\Delta > t) \to \mathbb{v}, \tag{7.15}$$

with $\mathbb{v} = \mathbb{v}_{cr}$.

Proof of Theorem 7.1. We prove the assertions in this order: the sufficiency of (7.12) followed by its necessity, then the uniqueness of a minimal QSD, the representation, and finally, the convergence.

Sufficiency. Suppose first that (7.12) holds. Apply Proposition 7.3 with $x_0 = x$, and let $d = d_0/2$, where d_0 is the positive constant obtained in the proposition. Then $\mathbf{X}^d$ satisfies the conditions of Theorem 2.1, and as a result possesses a unique minimal QSD, ν_{cr}, given by (2.2). Denote the minimal QSD for $\mathbb{X}$ obtained from ν_{cr} through the application of Proposition 7.4-2 by $\mathbb{v}_{cr}$.

Necessity. Suppose that $\mathbb{v}$ is a minimal QSD for $\mathbb{X}$. Let $d > 0$. Then from Proposition 7.4-1, $\mathbb{v}$ is a minimal QSD for $\mathbf{X}^d$ and therefore (2.1) holds for $\mathbf{X}^d$. Thus, $E_x[\exp(\mathbb{\lambda}_{cr} d(\tau_x^d \wedge \tau_\Delta^d))] < \infty$. But since $\mathbb{T}_x \wedge \mathbb{T}_\Delta \le d(\tau_x^d \wedge \tau_\Delta^d)$, it follows that (7.12) holds.

Uniqueness. Suppose that $\mathbb{v}$ and $\mathbb{v}'$ are minimal QSDs for $\mathbb{X}$. Then by Proposition 7.4-1, both are also minimal for $\mathbf{X}^d$, and by the uniqueness of a minimal QSD for $\mathbf{X}^d$, it follows that $\mathbb{v} = \mathbb{v}'$. We will, therefore, denote the unique minimal QSD for $\mathbb{X}$ by $\mathbb{v}_{cr}$.

Representation. As argued above, when it exists, $\mathbb{v}_{cr}$ is the unique minimal QSD for each of the processes $\mathbf{X}^{1/m}$ for all integer m large enough. We will denote all quantities associated with $\mathbf{X}^{1/m}$ with the superscript $\frac{1}{m}$. Write $\mu_x^{1/m}$ for the measure defined in Proposition 3.1 relative to the process $\mathbf{X}^{1/m}$ with $\alpha = e^{\mathbb{\lambda}_{cr}/m} = e^{\lambda_{cr}^{1/m}}$. Since $\mu_x^{1/m} p^{1/m} = e^{-\lambda_{cr}^{1/m}} \mu_x^{1/m}$, it follows from Proposition 1.1 and the uniqueness of the minimal QSD $\mathbb{v}_{cr}$ that

$$\mathbb{v}_{cr}(\cdot) = \frac{\mu_x^{1/m}(\cdot)}{\mu_x^{1/m}(\mathbf{1})}. \tag{7.16}$$

We will utilize this and a Riemann sum approximation to obtain the representation (7.13). Define

$$I_x(f) = E_x[\int_0^{\mathbb{T}_\Delta \wedge \mathbb{T}_x} \exp(\mathbb{\lambda}_{cr} s) f(\mathbb{X}_s) ds].$$

Then, a Riemann sum approximation and the dominated convergence theorem give

$$I_x(f) = \lim_{m\to\infty} E_x[\frac{1}{m} \sum_{0 \le n/m \le \mathbb{T}_\Delta \wedge \mathbb{T}_x} e^{\mathbb{\lambda}_{cr} n/m} f(\mathbb{X}_{n/m})]. \tag{7.17}$$

Because $\mathbf{X}^{1/m}$ is a snapshot of $\mathbb{X}$ at discrete intervals, $\tau_x^{1/m} \ge m\mathbb{T}_x$. Also, letting $A_m = \{\tau_x^{1/m} < m\mathbb{T}_x + 1\}$, we see that

$$P_x(A_m) \ge e^{-q_x/m} \underset{m\to\infty}{\to} 1.$$

On the event A_m, we can write the Riemann sum on the righthand of (7.17) as

$$\frac{1}{m}\left(\sum_{0\le n<\tau_x^{1/m}\wedge\tau_\Delta^{1/m}} e^{\lambda_{cr}^{1/m}n} f(X_n^{1/m}) + Ce^{\lambda_{cr}\mathbb{T}_\Delta\wedge\mathbb{T}_x}\right),$$

where $|C|$ is bounded by $\|f\|_\infty$. Taking expectations, this is equal to $\frac{1}{m}\mu_x^{1/m}(f)+O(1/m)$, and using (7.16), this is also equal to $\frac{1}{m}\mu_x^{1/m}(\mathbf{1})\mathbb{v}_{cr}(f)+O(\frac{1}{m})$. Since the expectation of the integral on the complement of A_m tends to zero as $m\to\infty$, we conclude that

$$I_x(f) = \lim_{m\to\infty}\frac{1}{m}\mu_x^{1/m}(\mathbf{1})\mathbb{v}_{cr}(f).$$

Use this with $f\equiv\mathbf{1}$, we have $I_x(\mathbf{1}) = \lim_{m\to\infty}\frac{1}{m}\mu_x^{1/m}(\mathbf{1})$, and so

$$\mathbb{v}_{cr}(f) = \frac{I_x(f)}{I_x(\mathbf{1})}.$$

The first expression for the QSD follows from a direct calculation of the denominator, and the second follows from Proposition 7.2-2. □

Next, we consider an analog of Theorem 2.2. For continuous-time processes, Ferrari, Kesten, Martinez, and Picco showed the convergence based on a renewal technique [10]. Martinez, San Martin, and Villemonais presented that the conditional distribution of the process converges exponentially fast in total variation norm to a unique QSD [19]. Convergence in total variation for processes satisfying strong mixing conditions was obtained using Fleming-Viot particle systems [7]. More recently, Champagnat and Villemonais stated a general criterion for uniform exponential convergence in total variation for absorbed Markov processes conditioned to survive [5, Assumption A]. They also provide analogous conditions involving Lyapunov functions, tailored to the situation where the convergence is non-uniform [6]. Our work considers the time-reversal at the quasi-stationarity of the absorbed Markov process, similar to the approach by Tough [25]. In particular, both Theorem 2.2 and Theorem 7.2 were inspired by and should be viewed as weaker versions of the main result in [19], but with a focus on the representation of the QSD rather than convergence to it. We provide more details on the difference in the statement of the theorem below.

Theorem 7.2. *Suppose that there exists some* $\bar{\lambda}>0$ *and a nonempty finite* $K\subsetneq S$

$$E_x[\exp(\bar{\lambda}\tau_\Delta)] = \infty \text{ and} \tag{7.18}$$

$$\sup_{x\notin K} E_x[\exp(\bar{\lambda}\mathbb{T}_\Delta\wedge\mathbb{T}_K)] < \infty. \tag{7.19}$$

Then

1. *The conditions of Theorem 7.1 hold. In particular, there exists a unique minimal QSD* $\mathbb{v}_{cr}$ *given by* (7.13).

2. *There are no other QSDs.*

3. (7.15) *holds for every finitely supported* μ *on* S.

If, in addition, there exists some $x_0 \in S$ *such that*

$$\inf_{x\in S} \frac{P_x(\mathbb{T}_{x_0} < \tau_\Delta)}{P_x(\mathbb{T}_\Delta > 1)} > 0 \tag{7.20}$$

then (7.15) *holds for any initial distribution* μ.

The main result of [19] gives the existence, uniqueness, and exponential convergence with an explicit bound on the total variation norm. The authors of that paperwork work under the following set of assumptions. First, they assume that S is countably infinite, Δ is a unique absorbing state, $P_x(\mathbb{T}_\Delta < \infty) = 1$ for all $x \in S$, as well as the following:

H1. There exists a finite non-empty $K \subsetneq S$ and a constant $c_1 > 0$ such that for all $t > 0$,
$$\inf_{x\in K} P_x(\mathbb{T}_\Delta > t) \le c_1 \sup_{x\in K} P_x(\mathbb{T}_\Delta > t).$$

H2. There exists a finite $K \subsetneq S$ and $x_0 \in K$ and constants $\lambda_0, c_2, c_3 > 0$ such that $\sup_{x\in S} E_x[\exp(\lambda_0(\mathbb{T}_K \wedge \mathbb{T}_\Delta))] \le c_2$ and $P_{x_0}(X_t \in K) \ge c_3 \exp(-\lambda_0 t)$ for all $t > 0$.

H3. There exists $x_0 \in K$ and a constant $c_4 > 0$ such that $\inf_{x\in S} P_x(\mathbb{X}_1 = x_0 \mid \mathbb{T}_\Delta > 1) \ge c_4$.

In part, **H1** and **H3** allow us to extend the discussion to processes not satisfying our irreducibility condition **HC-2**. We will now show that when S is irreducible, assumptions **H2** and **H3** imply all conditions of Theorem 7.2. Indeed, **H2** implies (7.18) and (7.19). Since $P_x(\mathbb{T}_{x_0} < \mathbb{T}_\Delta) \ge P_x(\mathbb{X}_1 = x_0)$, condition **H3** implies (7.20). The existence and uniqueness in [19] were established by obtaining uniform exponential bounds expressed in terms of the constants in **H1**,**H2** and **H3**, yet no formula for the QSD was obtained and relies on mathematical apparatus specifically developed to prove convergence. The focus of this work is on existence and representation, and our convergence result is obtained through an application of the ergodic positive recurrent and aperiodic discrete-time Markov chains, which is applicable to the reverse chain from Section 3.2.

Proof of Theorem 7.2. Let $m \in \mathbb{N}$ and let $d = 1/m$. The transition function p^d is automatically irreducible. Moreover, (7.18) and (7.19), yield the analogous statements for $\mathbf{X}^d$. Therefore $\mathbf{X}^d$ satisfies the first set of assumptions of Theorem 2.2. This leads to the first two numbered conclusions and the claimed convergence along the subsequence $t_n = nd, n \in \mathbb{Z}_+$. As (7.20) implies the

analogous condition (2.8) for $\mathbf{X}^d$, the final conclusion also holds along this subsequence. Note that this is valid for all positive d and that the unique QSD is independent of the choice of d.

It remains to prove the convergences along all sequences. Recall that $d = 1/m$, $m \in \mathbb{N}$. For every t, let $[t]_m = \lfloor tm \rfloor$, the corresponding "time" for the process $\mathbf{X}^{1/m}$. Observe that for every $t > 0$

$$P_\mu(X^{1/m}_{[t]_m} = y)e^{-q_y/m} \leq P_\mu(\mathbb{X}_t = y).$$

Or,

$$P_\mu(\mathbb{X}_t = y) \geq e^{-q_y/m} P_\mu(X^{1/m}_{[t]_m} = y) \underset{t\to\infty}{\sim} \mathbb{v}(y) P_\mu(\tau^{1/m}_\Delta > [t]_m).$$

If $\tau_\Delta > \frac{1}{m}[t]_m$, because $\tau^{1/m}_\Delta \geq m\mathbb{\tau}_\Delta$ and $[t]_m \leq tm$, it follows that $\{\tau^{1/m}_\Delta > [t]_m\} \supseteq \{m\mathbb{\tau}_\Delta > tm\}$. As a result,

$$\liminf_{t\to\infty} P_\mu(\mathbb{X}_t = y | \mathbb{\tau}_\Delta > t) \geq \mathbb{v}(y)e^{-q_y/m}.$$

Let $\nu_t = P_\mu(\mathbb{X}_t \in \cdot \,|\mathbb{\tau}_\Delta > t)$. Since m is arbitrary, $\liminf_{t\to\infty} \nu_t(y) \geq \nu(y)$. Fatuo's lemma gives that for any $A \subset S$, $\liminf_{t\to\infty} \nu_t(A) = \liminf_{t\to\infty} \sum_{x\in A} \nu_t(x) \geq \nu(A)$. Apply this to A^c to obtain

$$\nu(A^c) \leq \liminf_{t\to\infty} \sum_{x\in A^c} \nu_t(x) = 1 - \limsup_{t\to\infty} \sum_{x\in A} \nu_t(x).$$

Thus also $\limsup_{t\to\infty} \nu_t(A) \leq \nu(A)$, completing the proof. □

7.4 Finite MGF Regime

We begin with the analog of Theorem 2.4

Theorem 7.3. *Let $\lambda > 0$ be in the finite MGF regime. Then*

1. *If for some $\lambda' \in (0, \lambda)$, $\lim_{x\to\infty} E_x[\exp(\mathbb{\lambda}'\mathbb{\tau}_\Delta)] = \infty$ then there exists a QSD for $\mathbb{X}$ with absorption parameter λ'.*

2. *If $\sup_x E_x[\exp(\lambda\mathbb{\tau}_\Delta)] < \infty$, then there does not exist a QSD for $\mathbb{X}$ with absorption parameter λ.*

Proof. Fix some $d > 0$. We prove the two assertions in order of appearance:

1. Let $0 < \lambda' < \lambda$ satisfies $\lim_{x\to\infty} E_x[\exp(\lambda'\mathbb{\tau}_\Delta)] = \infty$. The first inequality in (7.9) gives $\lim_{x\to\infty} E_x[\exp(\lambda' d\tau^d_\Delta)] = \infty$ for the discrete process $\mathbf{X}^d$. By Theorem 2.4-1, we conclude that $\mathbf{X}^d$ has a QSD with absorption parameter λd, hence Proposition 7.4 implies $\mathbb{X}$ has a QSD with absorption parameter λ.

2. Given $\sup_x E_x[\exp(\lambda\mathbb{\tau}_\Delta)] < \infty$, the second inequality in (7.9) gives $\sup_x E_x[\exp(\lambda d\tau^d_\Delta)] < \infty$ for the discrete process $\mathbf{X}^d$. From Theorem 2.4-2, we conclude that $\mathbf{X}^d$ does not have a QSD with absorption parameter λd. Thus, from Proposition 7.4, we obtain that $\mathbb{X}$ has no QSD with absorption parameter λ.

□

We also have the following analog of Corollary 2.1

Corollary 7.1. *Let*

$$\lambda_0 = \inf\{\lambda \in (0, \lambda_{cr}) : \lim_{x\to\infty} E_x[\exp(\lambda \tau_\Delta)] = \infty\},$$

with the convention $\inf \emptyset = \infty$. *Then for every* $\lambda \in (\lambda_0, \lambda_{cr}]$ *there exists a QSD with absorption parameter* λ.

A special case is the main result of [10]. There, the authors proved the existence of a QSD under the assumption that for all $t > 0$,

$$\lim_{x\to\infty} P_x(\tau_\Delta \le t) \to 0.$$

This assumption implies $\lim_{x\to\infty} E_x[\exp(\lambda' \tau_\Delta)] = \infty$ for every $\lambda' > 0$, and therefore the corollary yields the existence of a QSD for every absorption parameter in the interval $(0, \lambda_{cr}]$.

Proof. For some $\lambda' > 0$, $\lim_{x\to\infty} E_x[\exp(\lambda' d \tau_\Delta^d)] = \infty$ and Corollary 2.1 guarantees that $\mathbf{X}^d$ has a QSD with absorption parameter in $(\lambda' d, \lambda_{cr} d]$. The result follows from Proposition 7.4. □

7.5 Martin Boundary

In this section, we provide a continuous-time version of Theorem 2.3. As in the previous sections, we adapt the results from the discrete setting.

Assume that $\lambda > 0$ is in the finite MGF regime. Fix any $d > 0$. Then the process $\mathbf{X}^d$ induces a Martin compactification, Definition 2.2. As in the definition, we write $K^{d\lambda}$, $\partial^{d\lambda} M$ and $(M^{d\lambda}, \rho^{d\lambda})$ for the corresponding Martin kernels, boundary and metric space. We also write $S^{d\lambda}$ for the elements in $\partial^{d\lambda} M$, which are QSDs for the transition function for $\mathbf{X}^d$ with absorption parameter $d\lambda$.

We need some preparations. First, we introduce the analogs of the kernels $K^{d\lambda}(\cdot,\cdot)$. For $x \in S$, define the kernel

$$\mathbb{K}^\lambda(x,y) = \frac{\int_0^\infty e^{\lambda s} P_x(\mathbf{X}_s = y) ds}{\int_0^\infty e^{\lambda s} P_x(\tau_\Delta > s) ds}. \tag{7.21}$$

Note that by our assumption that λ is in the finite MGF regime, both integrals are finite and nonzero. We make a connection with the kernels $K^{d\lambda}(\cdot,\cdot)$.

Lemma 7.1.

$$\mathbb{K}^\lambda(x,y) = \frac{\int_0^d e^{\lambda s} P_{K^{d\lambda}(x,\cdot)}(\mathbf{X}_s = y) ds}{\int_0^d e^{\lambda s} P_{K^{d\lambda}(x,\cdot)}(\tau_\Delta > s) ds}.$$

Proof. The numerator in (7.21) can be rewritten as

$$\begin{aligned}\sum_{m=0}^{\infty} e^{d\lambda m}\int_0^d e^{\lambda s}P_x(\mathbf{X}_{dm+s}=y)ds &= \sum_{m=0}^{\infty} e^{d\lambda m}\int_0^d e^{\lambda s}E_x[P_{X_m^d}(\mathbf{X}_s=y)]ds\\ &= \sum_{m=0}^{\infty}\sum_z e^{d\lambda m}P_x(X_m^d=z)\int_0^d e^{\lambda s}P_z(\mathbf{X}_s=y)ds\\ &= \sum_z G^{d\lambda}(x,z)\int_0^d e^{\lambda s}P_z(\mathbf{X}_s=y)ds\\ &= G^{d\lambda}(x,\mathbf{1})\int_0^d e^{\lambda s}P_{K^{d\lambda}(x,\cdot)}(\mathbf{X}_s=y)ds\end{aligned}$$

and similarly, the denominator is equal to $G^{d\lambda}(x,\mathbf{1})\int_0^d e^{\lambda s}P_{K^{d\lambda}(x,\cdot)}(\mathbb{T}_\Delta > s)ds$. □

Proposition 7.5. *Let $[\mathbf{x}]\in S^{d\lambda}$. Then*

1. *For every sequence $(x_n : n\in\mathbb{N})$ of elments in S which is in $[\mathbf{x}]$,*

$$\lim_{n\to\infty}\mathbb{K}^\lambda(x_n,y)=\frac{\int_0^d e^{\lambda s}P_{K^{d\lambda}([\mathbf{x}],\cdot)}(\mathbf{X}_s=y)ds}{\int_0^d e^{\lambda s}P_{K^{d\lambda}([\mathbf{x}],\cdot)}(\mathbb{T}_\Delta>s)ds},\ y\in S.$$

   Denote this limit by $\mathbb{K}^\lambda([\mathbf{x}],\cdot)$.

2. *$y\to\mathbb{K}^\lambda([\mathbf{x}],\cdot)$ is a QSD for $\mathbf{X}$ with absorption parameter λ.*

Proof. The first statement follows from applying the dominated convergence theorem to the identity in Lemma 7.1. Since $K^{d\lambda}([\mathbf{x}],\cdot)$ is a QSD for $\mathbf{X}^d$ with absorption parameter $d\lambda$, the second statement follows from Proposition 7.4-2 and the fact that $\mathbb{K}^\lambda([\mathbf{x}],\cdot)$ is a probability measure on S. □

We are ready to state the analog of Theorem 2.3.

Theorem 7.4. *Let $\lambda>0$ be in the finite MGF regime for $\mathbf{X}$. Let ν be a QSD for $\mathbf{X}$ with absorption parameter λ. Then, there exists a probability measure $\hat{F}_\nu$ on $\partial^{d\lambda}M$ satisfying $\hat{F}_\nu(S^{d\lambda})=1$ such that*

$$\nu(y)=\int\mathbb{K}^\lambda([\mathbf{x}],y)d\hat{F}_\nu([\mathbf{x}]).$$

Proof. By Proposition 7.4-1, ν is a QSD for $\mathbf{X}^d$ with absorption parameter $d\lambda$. Theorem 2.3 then gives a probability measure $\bar{F}_\nu$ with $\bar{F}_\nu(S^{d\lambda})=1$, satisfying

$$\nu(y)=\int K^{d\lambda}([\mathbf{x}],y)d\bar{F}_\nu([\mathbf{x}]).$$

Since ν is a QSD for $\mathbf{X}$ with absorption paramter λ, for every $s > 0$, $e^{\lambda s} P_\nu(\mathbf{X}_s = y) = \nu(y)$ and so $\nu(y) = \int_0^d e^{\lambda s} P_\nu(\mathbf{X}_s = y) ds$. An application of Fubini-Tonelli then gives

$$\begin{aligned}
\nu(y) &= \int \left(\frac{1}{d} \int_0^d e^{\lambda s} P_{K^{d\lambda}([\mathbf{x}],\cdot)}(\mathbf{X}_s = y) ds \right) d\bar{F}_\nu([\mathbf{x}]) \\
&= \int \mathbb{K}^\lambda([\mathbf{x}], y) \frac{\int_0^d e^{\lambda s} P_{K^{d\lambda}([\mathbf{x}],\cdot)}(\mathbb{T}_\Delta > s) ds}{d} \bar{F}_\nu([\mathbf{x}]) \\
&= \int \mathbb{K}^\lambda([\mathbf{x}], y) d\hat{F}_\nu([\mathbf{x}]),
\end{aligned}$$

where $\hat{F}_\nu$ is a measure absoultely continous with respect to $\bar{F}_\nu$, given by $\frac{d\hat{F}_\nu}{d\bar{F}_\nu}([\mathbf{x}]) = \frac{\int_0^d e^{\lambda s} P_{K^{d\lambda}([\mathbf{x}],\cdot)}(\mathbb{T}_\Delta > s) ds}{d}$. As ν and each $\mathbb{K}^\lambda([\mathbf{x}], \cdot)$ are probability measures, it follows that $\hat{F}_\nu$ is a probability measure too. $\square$

7.6 Example: QSDs for Birth and Death Process

In [26], Van-Doorn obtained all QSDs for birth and death processes on $\mathbb{Z}_+$, which are eventually absorbed at -1. This was done through a very detailed analysis of a spectral representation for the transition kernel of the process. Our work allows us to recover some of the main results using the general theory developed in earlier sections. We stress that the work in [26] contains many additional results, which we will not cover here, specifically regarding convergence.

In this section, we will drop **HC-1,2,3** to allow a more complete discussion consistent with the literature. We assume that $\mathbf{X}$ is a birth and death process on $\mathbb{Z}_+ \cup \{-1\}$, with birth and death rates $(\lambda_k : k \in \mathbb{Z}_+)$ and $(\mu_k : k \in \mathbb{Z}_+)$ respectively, which are all in $(0, \infty)$, and with -1 as a unique absorbing state. Letting

$$\pi_0 = 1, \pi_n = \prod_{j=1}^n \frac{\lambda_{j-1}}{\mu_j}, \ n \in \mathbb{N},$$

then

$$\sum_{n=0}^\infty \frac{1}{\lambda_n \pi_n} = \infty, \tag{7.22}$$

which is equivalent to $\mathbb{T}_\Delta < \infty$ a.s. from any initial distribution on $\mathbb{Z}_+$ [1, Chapter 8]. We will assume that (7.22) holds. This also implies that the process does not explode. Thus, **HC-1,2** automatically hold. Next, we introduce an array of random variables that are crucial for the analysis. Let $x \in \mathbb{Z}_+$ and $y \in \mathbb{Z}_+ \cup \{-1\}$ satisfying $x < y$ let $T_{x,y}$ be a random variable whose distribution is the same as τ_y under P_x. For each y, $T_{x,y} \preceq T_{x+1,y} \preceq \cdots$ and $T_{x,y+1} \preceq T_{x,y}$, where for two random variables X and Y, $X \preceq Y$ means that X is stochastically dominated by Y. Therefore, without loss of generality, we may assume all these RVs are realized in one probability space with the stochastic domination $\preceq$

replaced by a pointwise inequality $\leq$. With this, let $S_y = \lim_{x\to\infty} T_{x,y}$ and let $S = \sup S_y = S_{-1}$. This RV represents the passage time from $+\infty$ to $\Delta = -1$, and

$$E[S] = \sum_{n=1}^{\infty} \frac{1}{\lambda_n \pi_n} \sum_{i=n+1}^{\infty} \pi_i,$$

see [1, Chapter 8]. In order to apply our results, we need the following lemma.

Lemma 7.2. *1. Suppose $E[S] < \infty$. Then:*

(a) $\lambda_{cr} \in (0, \infty)$.

(b) $E_x[\exp(\lambda \mathbb{T}_\Delta)] < \infty$ if and only if $E[\exp(\lambda S)] < \infty$ if and only if $\lambda < \lambda_{cr}$.

(c) $E_x[\exp(\lambda_{cr} \mathbb{T}_\Delta \wedge \mathbb{T}_x)] < \infty$.

2. Suppose $E[S] = \infty$. Then $\lambda_{cr} = 0$ or $\lambda_{cr} > 0$. In the latter case, $\lim_{x\to\infty} E_x[\exp(\lambda \mathbb{T}_\Delta)] = \infty$ for all $\lambda \leq \lambda_{cr}$.

Proof. Suppose $E[S] < \infty$. Let $y \in \mathbb{Z}_+ \cup \{-1\}$, and define $h_y(t) = \sup_{x>y} P_x(\mathbb{T}_y > t) = P(S_y > t)$. Thus, $h_y(t) < E[S_y]/t$. Since $h_y(t+s) \leq h_y(t)h_y(s)$, it follows from Fekete's lemma that $\lim_{t\to\infty} \frac{\ln h_y(t)}{t} = \inf_{t>0} \frac{\ln h_y(t)}{t} = -c_y$. Thus, for every $t > 0$

$$-c_y \leq \frac{\ln h_y(t)}{t} \leq \ln(E[S_y]/t)/t. \tag{7.23}$$

And by choosing $t > E[S_y]$, the righthand side of (7.23) is strictly negative, which guarantees $c_y > 0$. Since $S_y = \sum_{x=y}^{\infty} T_{x+1,x}$, it follows from dominated convergence that $E[S_y] \to 0$ as $y \to \infty$, using this in (7.23) shows that $c_y \to \infty$ as $y \to \infty$. Also, $c_y < \infty$ because S_y stochastically dominates $T_{y+1,y}$ which is $\mathrm{Geom}(\lambda_{y+1} + \mu_{y+1})$. From the definition of c_y, $E[\exp(\lambda S_y)] < \infty$ if $\lambda < c_y$ and $= \infty$ if $\lambda > c_y$. Also $S = S_{-1}$ is the independent sum of S_y and $T_{y,-1}$. Thus for any $\lambda > 0$,

$$E[\exp(\lambda S)] = E[\exp(\lambda S_y)] E_y[\exp(\lambda \mathbb{T}_\Delta)].$$

If the lefthand side is finite, then both terms on the right-hand side are finite. If the lefthand side is infinite, by choosing y such that $c_y > \lambda$, the first term on the right-hand side is finite, and therefore the second term on the right-hand side is infinite. This also shows that $\lambda_{cr} = c_{-1}$. Finally, if $E[\exp(\lambda_{cr} S)] < \infty$, the family of RVs $(T_{y,-1} : y \in \mathbb{Z}_+)$ is uniformly integrable. This violates Proposition 2.4, whose proof is valid in the continuous time setting with only the change of notation. Since $\inf_y P_y(\mathbb{T}_y < \mathbb{T}_\Delta) > 0$, $E_x[\exp(\lambda_{cr} \mathbb{T}_\Delta \wedge \mathbb{T}_x)] < \infty$ as a result of Proposition 2.5, which is also valid in the present setting with the obvious adaptations.

Next, consider the case $E[S] = \infty$. If $\lambda_{cr} > 0$, mononote convergence gives that for any $\lambda \in (0, \lambda_{cr})$, $\lim_{x\to\infty} E_x[\exp(\lambda \mathbb{T}_\Delta)] = E[\exp(\lambda S)] = \infty$. □

With the lemma, we can prove the following characterization and description of QSDs for Birth and Death processes. This result is equivalent to [26,

Theorem 3.2], which was the first to characterize and describe all QSDs for Birth and Death processes through spectral analysis of the transition kernels and corresponding orthogonal polynomials.

Theorem 7.5 (Theorem 3.2, [26]). *1. Suppose $E[S] < \infty$. Then $\lambda_{cr} > 0$, and there exists a unique QSD, which is also minimal.*

2. Suppose that $E[S] = \infty$. Then either $\lambda_{cr} = 0$ and there are no QSDs or $\lambda_{cr} > 0$ and for every $\lambda \in (0, \lambda_{cr}]$ there exists a QSD with absorption parameter λ.

3. When exists, a QSD with absorption parameter $\lambda > 0$ is unique and given by the formula

$$\nu_\lambda(y) = \frac{\lambda}{q_y - \lambda} \frac{1}{E_y[\exp(\lambda \tau_\Delta), \tau_\Delta < \tau_y]}, \ y \in S. \tag{7.24}$$

We comment that letting $y = 0$ in (7.24), a straighforward calculation revelas that

$$\nu_\lambda(0) = \frac{\lambda}{\mu_0}$$

(this can be independently obtained from [26, equation (3.4)]). Thus, a necessary condition for the existence of a QSD with absorption parameter λ is $\lambda < \mu_0$. As $\nu_\lambda(-1) = 0$, these two initial values can be used to solve the system of difference equations resulting from (7.5).

We also comment that the argument leading to (7.24) is valid for any chain which is downward skip-free and that a simple calculation shows that when $E[S] < \infty$ and $\lambda \in (0, \lambda_{cr})$, the pointwise limit of $(\mathbb{K}^\lambda(n, \cdot) : n \in \mathbb{N})$ along any convergent subsequence can be normalized to be a probability measure on S which satisfies the system of difference equations resulting from (7.5) but is not a QSD.

Proof. If $E[S] < \infty$, Lemma 7.2-1 guarantees that the conditions of Theorem 7.1 hold. This yields the existence and uniqueness of a minimal QSD. For $\lambda < \lambda_{cr}$, $\sup_x E_x[\exp(\lambda \tau_\Delta)] \leq E[\exp(\lambda S)] < \infty$ and therefore Theorem 7.3-2 shows that no other QSDs exist.

If $E[S] = \infty$ and $\lambda_{cr} > 0$, Lemma 7.2-2 and Corollary 7.1 give the existence of QSDs for each of the absorption parameters in the range $(0, \lambda_{cr}]$. The remaining case is $E[S] = \infty$ and $\lambda_{cr} = 0$. In this case, no QSDs exist, as this will lead to a violation of (1.5) for each of the discretized processes.

It remains to establish the representation formula. The formula holds in the infinite MGF regime due to (7.13). Suppose that ν is a QSD with absorption parameter λ in the finite MGF regime, then Theorem 7.4 implies that it is in the convex hull of $\mathbb{K}^\lambda([\mathbf{x}], \cdot)$ where $[\mathbf{x}]$ ranges over $S^{d\lambda}$. In particular, the latter is not empty. We will show that for any sequence $(x_n : n \in \mathbb{N})$ satisfying $\lim_{n\to\infty} x_n = \infty$, $\mathbb{K}^\lambda(x_n, \cdot)$ converges pointwise to a limit independent of the sequence, given by the formula in the statement of the theorem. As by assumption $S^{d\lambda}$ is not empty, this guarantees that $S^{d\lambda}$ has a unique element equal to that limit.

Indeed,

$$\mathbb{K}^\lambda(x_n, y) = \frac{E_{x_n}[\exp(\lambda \mathbb{T}_y)] I_\lambda(y)}{E_{x_n}[\int_0^{\mathbb{T}_\Delta} e^{\lambda s} ds]},$$

where $I_\lambda(y) = \int_0^\infty e^{\lambda s} P_y(\mathbf{X}_s = y) ds$. The denominator is equal to $\frac{1}{\lambda}\left(E_{x_n}[\exp(\lambda \mathbb{T}_\Delta)] - 1\right)$. Lemma 7.2-2 gives that $\lim_{n\to\infty} E_{x_n}[\exp(\lambda \mathbb{T}_\Delta)] = \infty$. By the strong Markov property, $E_{x_n}[\exp(\lambda \mathbb{T}_\Delta)] = E_{x_n}[\exp(\lambda \mathbb{T}_y)] E_y[\exp(\lambda \mathbb{T}_\Delta)]$, and therefore the denominator is asymptotically equivalent to $E_{x_n}[\exp(\lambda \mathbb{T}_y)] \lambda^{-1} E_y[\exp(\lambda \mathbb{T}_\Delta)]$, resulting in

$$\lim_{n\to\infty} \mathbb{K}^\lambda(x_n, y) = \frac{\lambda I_\lambda(y)}{E_y[\exp(\lambda \mathbb{T}_\Delta)]}.$$

We evaluate $I_\lambda(y)$. Let $J = \inf\{t \in \mathbb{R}_+ : \mathbf{X}_t \neq \mathbf{X}_{t-}\}$, the time of the first jump. Under P_y, $J \sim \mathrm{Exp}(q_y)$. Breaking the integral in the definition of $I_\lambda(y)$ we have

$$I_\lambda(y) = E_y[\int_0^J e^{\lambda s} ds] + E_y[\exp(\lambda \mathbb{T}_y), \mathbb{T}_y < \mathbb{T}_\Delta] I_\lambda(y),$$

and so

$$I_\lambda(y) = \frac{E_y[\exp(\lambda J)] - 1}{\lambda(1 - E_y[\exp(\lambda \mathbb{T}_y), \mathbb{T}_y < \mathbb{T}_\Delta])},$$

which in turn gives

$$\begin{aligned}\lim_{n\to\infty} \mathbb{K}^\lambda(x_n, y) &= \frac{E_y[\exp(\lambda J)] - 1}{E_y[\exp(\lambda \mathbb{T}_\Delta)](1 - E_y[\exp(\lambda \mathbb{T}_y), \mathbb{T}_y < \mathbb{T}_\Delta])} \\ &= \frac{\lambda}{q_y - \lambda} \frac{1}{E_y[\exp(\lambda \mathbb{T}_\Delta), \mathbb{T}_\Delta < \mathbb{T}_y]}\end{aligned}$$

□

8 Analysis of Minimal QSDs for a One-Parameter Family

8.1 Main Results

In this section, we study in detail the minimal QSDs for one-parameter family processes, all of which are special cases of the rooted tree of Section 6.5. We show that for some values of the parameter, λ_{cr} is in the infinite MGF regime, and for others, it is in the finite MGF regime. Moreover, when the latter alternative holds, the minimal QSDs form a two-dimensional convex cone (Proposition 8.3), and although the chain is aperiodic, the limiting conditional probabilities exhibit periodicity (Theorem 8.1).

The model is essentially two birth and death chains glued at zero. Fix $q \in (\frac{1}{2}, 1)$, let $\delta \in (0, q]$, and set $r = q - \delta$. Consider the Markov chain $\mathbf{X} = (X_n : n \in \mathbb{Z}_+)$ on $\mathbb{Z} \cup \{\Delta\}$ with transitions as in Figure 3. Let λ_{cr} be the critical absorption parameter for $\mathbf{X}$, for $x \in \mathbb{Z} \cup \{\Delta\}$ define $\tau_x = \inf\{n \in \mathbb{Z}_+ : X_n = x\}$ and write P_x for the probability of $\mathbf{X}$.

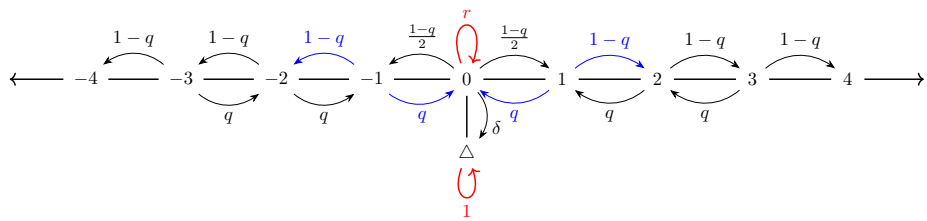

Figure 3: Transition Probabilities Diagram

We first examine the dependence of the critical absorption parameter λ_{cr} on δ. To do that, let λ_0 denote the critical absorption parameter for the system restricted to $\mathbb{Z}_+$ (or equivalently $-\mathbb{Z}_+$) and absorbed when hitting 0.

Proposition 8.1. *1.* $\exp(-\lambda_0) = 2\sqrt{q(1-q)}$

2. Let $\delta_{cr} = \sqrt{q}(\sqrt{q} - \sqrt{1-q})$. *Then*

$$\exp(-\lambda_{cr}) = \begin{cases} q - \delta + \frac{q(1-q)}{q-\delta} & \delta \in (0, \delta_{cr}) \\ \exp(-\lambda_0) & \delta \in [\delta_{cr}, q] \end{cases}.$$

Moreover,

$\delta \in$	$(0, \delta_{cr})$	$\{\delta_{cr}\}$	$(\delta_{cr}, q]$
λ_{cr}	$< \lambda_0$	$= \lambda_0$	
$E_0[\exp(\lambda_{cr}\tau_\Delta)]$	$= \infty$		$< \infty$

(8.1)

In particular, λ_{cr} is in the finite or infinite MGF regime according to the value of δ.

We now present the results on the minimal QSDs according to the value of δ.

Proposition 8.2. *Let* $\delta \le \delta_{cr}$. *Then*

1. λ_{cr} *is in the infinite MGF regime and condition* (2.1) *holds. In particular,* $\mathbf{X}$ *has a unique minimal QSD given by*

$$\nu_{cr}(x) = \frac{e^{\lambda_{cr}} - 1}{E_x[\exp(\lambda_{cr}\tau_\Delta), \tau_\Delta < \tau_x]}, \ x \in \mathbb{Z}.$$

2. *Condition* (2.3) *holds if and only if* $\delta < \delta_{cr}$.

Next, we discuss the case $\delta > \delta_{cr}$. It follows from Corollary 2.1 that for every $\lambda \in (0, \lambda_{cr}]$, there exists a QSD with absorption parameter λ. For real y, we adopt the convention $y_+ = \max(y, 0)$, $y_- = (-y)_+ = \max(-y, 0)$. Let

$$\rho = \sqrt{\frac{1-q}{q}} \in (0, 1).$$

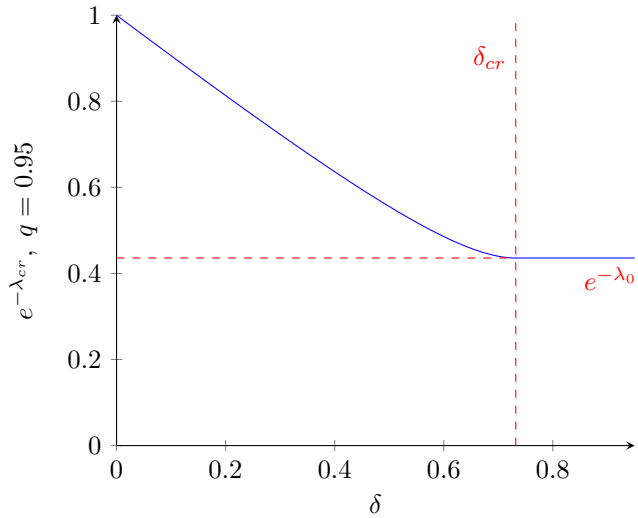

Figure 4: absorption parameter dependence on δ when $q = 0.95$

With this,

$$\delta_{cr} = q - q\rho.$$

Since $r + \delta = q$, we have that $r < q - \delta_{cr} = q\rho = \sqrt{q(1-q)}$. As a result, in the $\delta > \delta_{cr}$ regime, we have

$$\begin{aligned} r &= \alpha\sqrt{q(1-q)}, \text{ for some } \alpha \in [0,1), \\ \delta &= q - \alpha\sqrt{q(1-q)} = q(1-\alpha\rho). \end{aligned}$$

Proposition 8.3. *Let $\delta > \delta_{cr}$. Then the set of minimal QSDs S^λ is a two-dimensional convex cone spanned by $\{\mu_+^{\lambda_0}, \mu_-^{\lambda_0}\}$, where*

$$\mu_\pm^{\lambda_0}(y) = \frac{(1-\rho)^2}{1-\rho\alpha}\rho^{|y|} \times \begin{cases} 1 & y = 0 \\ \frac{1}{2} + (1-\alpha)y_\pm & y \in \mathbb{Z} - \{0\} \end{cases} \tag{8.2}$$

Define the minimal QSD μ^{λ_0},

$$\mu^{\lambda_0}(y) = \frac{1}{2}(\mu_+^{\lambda_0} + \mu_-^{\lambda_0})(y) = \frac{(1-\rho)^2}{1-\rho\alpha}\rho^{|y|} \times \begin{cases} 1 & y = 0 \\ \frac{1}{2} + \frac{1}{2}(1-\alpha)|y| & y \in \mathbb{Z} - \{0\}. \end{cases}$$

We have the following result

Theorem 8.1. *Suppose $\delta > \delta_{cr}$. For $x, y \in \mathbb{Z}$, define*

$$\begin{aligned} h(x) &= \frac{(1-\alpha)|x|}{1 + (1-\alpha)|x|} \\ \kappa(x) &= sgn(x)h(x) \end{aligned} \tag{8.3}$$

Then

1. $\lim_{n\to\infty} P_x(X_{2n} = y|\tau_\Delta > 2n) = \mu^{\lambda_0}(y) + \mathbf{1}_{2\mathbb{Z}}(y-x)\kappa(x)\dfrac{\mu_+^{\lambda_0}(y) - \mu_-^{\lambda_0}(y)}{2}$.

2. $\lim_{n\to\infty} P_x(X_{2n+1} = y|\tau_\Delta > 2n+1) = \mu^{\lambda_0}(y) + \mathbf{1}_{2\mathbb{Z}+1}(y-x)\kappa(x)\dfrac{\mu_+^{\lambda_0}(y) - \mu_-^{\lambda_0}(y)}{2}$.

In other words, the conditional probabilities along each of the subsequences of even times and odd times converge to explicitly identifiable limits. On each such sequence, the limit is not a QSD, with one notable exception, $x = 0$, where the limit is μ^{λ_0}. When $x \neq 0$, the restriction of the limit to even integers and the restriction to the odd integers are each given by convex combinations of the QSDs in Proposition 8.3, one of which is μ^{λ_0}.

8.2 Proof of Proposition 8.1

Let $\mathbf{Y} = (Y_n : n \in \mathbb{Z}_+)$ be the Birth and Death process with transitions as in Figure 5. Clearly, this transition function satisfies Assumption **HD-1, HD-2** and **HD-3**.

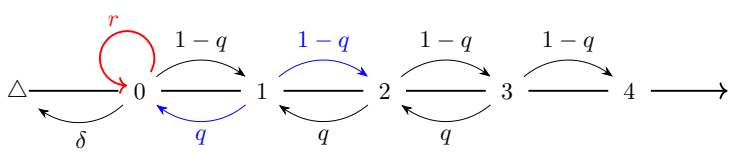

Figure 5: The Birth & Death Process $\mathbf{Y}$

For $y \in \mathbb{Z}_+ \cup \{\Delta\}$, let $\tau_y^Y = \inf\{n \in \mathbb{N} : Y_n = y\}$. We also need the induced process $\mathbf{Y}^0$, defined as follows:

$$Y_n^0 = Y_n \mathbf{1}_{\{\tau_0^Y > n\}}.$$

That is, $\mathbf{Y}^0$ is $\mathbf{Y}$, absorbed at 0. Note that the distribution of $\mathbf{Y}$ starting from x coincides with the distribution of $|\mathbf{X}|$ starting from $x \in \mathbb{Z}_+$, and that the distribution of $\mathbf{Y}^0$ starting from $x \in \mathbb{N}$ coincides with the distribution of $|\mathbf{X}| + 1$ with $\alpha = 0$ (hence $\delta = q$), starting from $x + 1$. Thus, we can reduce the discussion to the auxiliary birth and death process from Figure 5.

Proof of Proposition 8.1. Starting from $x \geq 1$, $\mathbf{Y}$ must pass through 0 in order to get to Δ. Therefore for all values of δ, $\lambda_{cr} \leq \lambda_0$.

Step 1: Calculation of λ_0. Define $f(x) = f(x, \lambda) = E_x[\exp(\lambda\tau_0)], x \in \mathbb{N}$, and condition on the first step from x and spatial homogeneity of the process to obtain

$$\begin{aligned} f(1) &= e^\lambda[q + (1-q)f(2)] \text{ and} \\ &= e^\lambda[q + (1-q)f^2(1)] \end{aligned}$$

Hence we have a quadratic equation for $f(1) = f$:

$$e^{\lambda}(1-q)f^2 - f + e^{\lambda}q = 0$$

The quadratic formula gives

$$f_{1,2}(1) = \frac{1 \pm \sqrt{1 - 4e^{2\lambda}(1-q)q}}{2e^{\lambda}(1-q)} \tag{8.4}$$

Suppose $\lambda < \lambda_{cr}$. Then since $f(1)$ is real-valued, we must have $e^{\lambda} \leq \dfrac{1}{2\sqrt{(1-q)q}}$. Also, since in this region, the function $\lambda \to f(1,\lambda)$ is increasing, we have $f(1,\lambda) = \dfrac{1 - \sqrt{1 - 4e^{2\lambda}(1-q)q}}{2e^{\lambda}(1-q)}$. With this, we conclude that

$$e^{\lambda_0} = \frac{1}{2\sqrt{(1-q)q}}, \quad E_1[\exp(\lambda_0 \tau_0)] = f(1,\lambda_0) = \sqrt{\frac{q}{1-q}} < \infty. \tag{8.5}$$

Step 2. Calculation of λ_{cr}. Similar first-step analysis applied to $v = v(\lambda) = E_0[\exp(\lambda \tau_\Delta)]$ gives

$$v = e^{\lambda}[\delta + rv + (1-q)fv],$$

where here $f = f(1,\lambda)$. That is,

$$v[(1 - e^{\lambda}(q - \delta + (1-q)f)] = e^{\lambda}\delta.$$

Hence

$$v(\lambda) = \frac{e^{\lambda}\delta}{1 - e^{\lambda}(q - \delta + (1-q)f)} \tag{8.6}$$

Using the fact that u is non-decreasing as a function of λ, u is finite if and only if $e^{\lambda}(q - \delta + (1-q)f) < 1$. This implies

$$\lambda_{cr} = \sup\{\lambda > 0 : e^{\lambda}(q - \delta + (1-q)f) < 1\} \tag{8.7}$$

(here we take $\sup \emptyset = 0$). Since we already know that $\lambda_{cr} \leq \lambda_0$, this supremum is finite, and we only need to consider $\lambda \leq \lambda_0$. The supremum is clearly non-decreasing in δ. We examine the two extreme values for δ:

- When $\delta = 0$, $e^{\lambda}(q + (1-q)f) > 1$ for all $\lambda > 0$ and therefore the supremum is zero.
- For $\delta = q$, $e^{\lambda_0}(0 + (1-q)f(1,\lambda_0)) = \frac{1}{2} < 1$, and so $\lambda_{cr} = \lambda_0$.

From (8.7), $\lambda_{cr} = \lambda_0$ if and only if

$$e^{\lambda_0}(q - \delta + (1-q)f(1,\lambda_0)) \leq 1,$$

or equivalently,

$$\delta \geq q + (1-q)f(1,\lambda_0) - e^{-\lambda_0} = q - \sqrt{q(1-q)} = \sqrt{q}(\sqrt{q} - \sqrt{1-q}) = \delta_{cr}.$$

By (8.6), $u(\lambda_0) = \infty$ when $\delta = \delta_{cr}$ and $u(\lambda_0) < \infty$ when $\delta > \delta_{cr}$.

It remains to find λ_{cr} when $\delta < \delta_{cr}$. Because $\delta < \delta_{cr}$, we have $e^{\lambda_0}[q - \delta + (1-q)f(\lambda_0)] > 1$, and therefore λ_{cr} is the unique solution in $(0, \lambda_0)$ to

$$e^{\lambda}[q - \delta + (1-q)f(\lambda)] = 1,$$

and from (8.7), $u(\lambda_{cr}) = \infty$. Since $f = \frac{1-\sqrt{1-4e^{2\lambda}(1-q)q}}{2e^{\lambda}(1-q)}$ the equation to be solved becomes

$$e^{\lambda}[q - \delta + \frac{1-\sqrt{1-4e^{2\lambda}(1-q)q}}{2e^{\lambda}}] = 1 \tag{8.8}$$

After simplifying (8.8), we obtain the expression in the statement of the proposition, completing the proof. □

8.3 Proof of Proposition 8.2

Proof of Proposition 8.2. In the case $\delta < \delta_{cr}$, Proposition 8.1 gives $E_x[\exp(\lambda_{cr}\tau_\Delta)] = \infty$. We examine

$$E_x[\exp(\lambda_{cr}\tau_\Delta), \tau_\Delta < \tau_x].$$

The key is to show

$$E_0[\exp(\lambda_{cr}\tau_\Delta), \tau_\Delta < \tau_0] < \infty. \tag{8.9}$$

We consider two cases:

1. $\delta < \delta_{cr}$. In this case, $\lambda_{cr} < \lambda_0$, by Definition 1.2, we have

$$E_0[\exp(\lambda_{cr}\tau_\Delta), \tau_\Delta < \tau_0] < \infty$$

2. $\delta = \delta_{cr}$. In this case $\lambda_{cr} = \lambda_0$, condition on the first step from $x = 0$ and spatial homogeneity of the process yields

$$E_0[\exp(\lambda_{cr}\tau_\Delta), \tau_\Delta < \tau_0] = \delta e^{\lambda_{cr}} < \infty.$$

Therefore, by (8.9) and the irreducibility we conclude for $x \in S$,

$$E_x[\exp(\lambda_{cr}\tau_\Delta), \tau_\Delta < \tau_x] < \infty,$$

which satisfies the necessary and sufficient condition in Theorem 2.1 and hence we obtain a unique minimal QSD given by (2.2).

Next, for $\lambda \leq \lambda_0$, we define $u(\lambda) = E_0[\exp(\lambda\tau_0), \tau_0 < \tau_\Delta]$. Conditioning on the first step

$$u(\lambda) = (q - \delta)e^{\lambda} + (1-q)e^{\lambda}f(\lambda),$$

where $f(\lambda) = E_1[\exp(\lambda\tau_0), \tau_0 < \tau_\Delta]$. The proof of Proposition 8.1 gives $f(\lambda)$ (see the discussion below (8.4)), and so

$$u(\lambda) = (q - \delta)e^{\lambda} + \frac{1-\sqrt{1-4q(1-q)e^{2\lambda}}}{2}.$$

We have two cases:

1. $\delta < \delta_{cr}$. In this case, $\lambda_{cr} < \lambda_0$, the left derivative of $u(\lambda)$ at λ_{cr} is finite, and therefore $E_0[\exp(\lambda_{cr}\tau_0)\tau_0, \tau_0 < \tau_\Delta] < \infty$.

2. $\delta = \delta_{cr}$. In this case, $\lambda_{cr} = \lambda_0$, the left derivative of $u(\lambda)$ at λ_0 is infinite, and therefore $E_0[\exp(\lambda_0\tau_0)\tau_0, \tau_0 < \tau_\Delta] = \infty$.

Hence, by Proposition 1.2-2, Propostion 3.3 and the irreducibility we conclude for $x \in S$,

$$E_x[\exp(\lambda_{cr}\tau_x)\tau_x, \tau_x < \tau_\Delta] < \infty$$

if and only if $\delta < \delta_{cr}$, completing the proof. □

8.4 Proof of Proposition 8.3

Throughout this section, we assume $\delta > \delta_{cr}$, a case in which the critical absorption parameter for all of the three processes $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Y}^0$ is the same and is equal to λ_0. Moreover, λ_0 is in the finite MGF regime for all three processes by Proposition 8.1.

Lemma 8.1. *1. $\mathbf{Y}^0$ has a unique minimal QSD $\nu^{\lambda_0,0}$ given by*

$$\nu^{\lambda_0,0}(y) = (1-\rho)^2 y\rho^{y-1},\ y \in \mathbb{N}$$

2. $\mathbf{Y}$ has a unique minimal QSD ν^{λ_0} given by

$$\nu^{\lambda_0}(y) = \frac{(1-\rho)^2}{1-\rho\alpha}(1+(1-\alpha)y)\rho^y,\ y \in \mathbb{Z}_+.$$

Because λ_0 is in the finite MGF regime and both $\mathbf{Y}$ and $\mathbf{Y}^0$ satisfy condition 2.11, it follows from Proposition 2.3-2, that each has unique minimal QSD, and so the identities above can be verified by direct computation. Alternatively, a proof of the first can be found in [20, Proposition 6] or [18, Theorem 5.1], and a proof of the second can be found in [24].

Proof of Proposition 8.3. Suppose ν is a minimal QSD for $\mathbf{X}$. Define

$$\alpha(y) = \begin{cases} \nu(y) + \nu(-y) & y \neq 0 \\ \nu(0) & y = 0 \end{cases}$$
$$\beta(y) = \nu(y) - \nu(-y).$$

A straightforward verification whose details are omitted reveals that the restriction of α to $\mathbb{Z}_+$ satisfies the system of equations (1.6) for the transition function for $\mathbf{Y}$ with $\lambda = \lambda_0$. The set of solutions to this equation is one dimensional (the equation for $j = 0$ shows that $\nu(0)$ uniquely determines $\nu(1)$, and these two determine $\nu(j)$ for all $j = 1, \dots$), and as ν^{λ_0} is a strictly positive solution to the equations, we have that for some $c > 0$

$$\alpha(y) = c\nu^{\lambda_0}(|y|),\ y \in \mathbb{Z}.$$

Summing over $y \in \mathbb{Z}_+$, the righthand side gives c, and the lefthand side gives 1. Therefore $c = 1$. An identical argument leads to the conclusion that there exists some constant c such that the restriction of β to $\mathbb{N}$ is $c\nu^{\lambda_0,0}$, and as a result

$$\beta(y) = c \cdot \mathrm{sgn}(y)\nu^{\lambda_0,0}(|y|),\ y \in \mathbb{Z}.$$

Since $\nu(0) = \alpha(0)$ and for all other y, $\nu(y) = \frac{\alpha(y)+\beta(y)}{2}$, we have

$$\nu(y) = \begin{cases} \nu^{\lambda_0}(0) & y = 0 \\ \frac{1}{2}\left(\nu^{\lambda_0}(|y|) + c \cdot \mathrm{sgn}(y)\nu^{\lambda_0,0}(|y|)\right) & y \neq 0 \end{cases}$$

For all values of c, the function on the righthand side sums to 1. It also satisfies (1.6) for the transition function for $\mathbf{X}$ with $\lambda = \lambda_0$. It is nonnegative and, therefore, a minimal QSD if and only if

$$\inf_{y \in \mathbb{N}} \nu^{\lambda_0}(y) - |c|\nu^{\lambda_0,0}(y) \geq 0.$$

Using the explicit formulas for the two QSDs in the inequality, we obtain

$$|c| \leq \inf_{y \in \mathbb{N}} \frac{\rho(1 + (1-\alpha)y)}{(1-\rho\alpha)y} = \frac{\rho(1-\alpha)}{1-\rho\alpha}.$$

Therefore ν is necessarily a convex combination of the two QSDs obtained by choosing $c = \pm\frac{\rho(1-\alpha)}{1-\rho\alpha}$, which we respectively denote by $\mu^{\lambda}_{\pm}$, and are given by

$$\mu^{\lambda_0}_{\pm}(y) = \frac{1}{2}(1 + \delta_0(y))\nu^{\lambda_0}(|y|) \pm \frac{1}{2}\frac{\rho(1-\alpha)}{1-\rho\alpha}\mathrm{sgn}(y)\nu^{\lambda_0,0}(|y|). \tag{8.10}$$

Hence Lemma 8.1-1 and -2 give

$$\mu^{\lambda_0}_{\pm}(y) = \frac{(1-\rho)^2}{1-\rho\alpha}\rho^{|y|} \times \begin{cases} 1 & y = 0 \\ \frac{1}{2} + (1-\alpha)y_{\pm} & y \in \mathbb{Z} - \{0\} \end{cases}$$

□

8.5 Proof of Theorem 8.1

To distinguish between probabilities and expectations for $\mathbf{X}$ and $\mathbf{Y}^0$, we denote the distribution of $\mathbf{Y}$ starting from (state or distribution) $\cdot$ by $Q_\cdot$. The proof of the theorem requires the following results that show the asymptotic distribution of $\mathbf{Y}^0$ and $\mathbf{Y}$, respectively.

Lemma 8.2. *1. Suppose $x, y \in \mathbb{N}$. Then*

$$\begin{aligned} Q_x(Y_n = y, \tau_0^Y > n) &\sim e^{-\lambda_0 n}\sqrt{\frac{8}{\pi n^3}}x\rho^{-x}y\rho^y\mathbf{1}_{\{y \in x-n+2\mathbb{Z}\}} \\ &= e^{-\lambda_0 n}\sqrt{\frac{8}{\pi n^3}}x\rho^{-x}\frac{\mathbf{1}_{\{y \in x-n+2\mathbb{Z}\}}\rho}{(1-\rho)^2}\nu^{\lambda_0,0}(y). \end{aligned}$$

2. Suppose $x \in \mathbb{N}$. Then

$$Q_x(\tau_0^Y > n) \sim e^{-\lambda_0 n}\sqrt{\frac{8}{\pi n^3}}x\rho^{-x}\frac{\rho}{(1-\rho^2)^2}\begin{cases}2\rho & n \in x+2\mathbb{Z}\\ 1+\rho^2 & n \in x+2\mathbb{Z}-1\end{cases}$$

Lemma 8.3. *Suppose $\delta > \delta_{cr}$ and let $r = \alpha\sqrt{q(1-q)}$ for some $\alpha \in [0,1)$. Then for $x, y \in \mathbb{Z}_+$*

1. $Q_x(Y_n = y, \tau_\Delta^Y > n) \sim \dfrac{(1-\rho)^2}{1-\alpha\rho}\rho^y\left(1+(1-\alpha)y\right)Q_x(\tau_\Delta^Y > n)$.

2.

$$Q_x(\tau_\Delta^Y > n) \sim e^{-\lambda_0 n}\sqrt{\frac{8}{\pi n^3}}\frac{(1-\alpha\rho)\rho^{-x}}{[(1-\rho^2)(1-\alpha^2)]^2}$$
$$\times\begin{pmatrix}1+\rho^2\\ 2\rho\end{pmatrix}\cdot\left[J_2^{x+n}\left(\begin{pmatrix}1+\alpha^2\\ 2\alpha\end{pmatrix}+(1-\alpha^2)x\begin{pmatrix}1\\ \alpha\end{pmatrix}\right)\right]$$

Where J_2 is the 2×2 exchange matrix.

As also shown in [24], Lemma 8.2 and Lemma 8.3 imply that the limiting conditional distributions exist and are given by:

Corollary 8.1. *1.*

$$Q_x(Y_n = y|\tau_0^Y > n) \sim \nu^{\lambda_0,0}(y)\times\begin{cases}\mathbf{1}_{2\mathbb{N}}(y)\frac{(1+\rho)^2}{2\rho} & n \in x+2\mathbb{Z}\\ \mathbf{1}_{2\mathbb{N}-1}(y)\frac{(1+\rho)^2}{1+\rho^2} & n \in x+2\mathbb{Z}-1\end{cases}$$

2. Suppose $\delta > \delta_{cr}$. Then

$$Q_x(Y_n = y|\tau_\Delta^Y > n) \sim \frac{(1-\rho)^2}{1-\rho\alpha}\rho^y\left(1+(1-\alpha)y\right) \sim \nu^{\lambda_0}(y).$$

We will give the proof of Lemma 8.2 and Lemma 8.3 in Section 8.6.

Proof of Theorem 8.1. Obsevre that for every $x \in \mathbb{Z}$, the distribution of $|\mathbf{X}|$ under P_x coincides with the distribution of $\mathbf{Y}$ under $Q_{|x|}$, and in particular,

$$P_x(X_n = 0|\tau_\Delta > n) = Q_{|x|}(Y_n = 0|\tau_\Delta^Y > n) \to \nu^{\lambda_0}(0).$$

Next, consider $P_x(X_n = y|\tau_\Delta > n)$ for general $y \in \mathbb{Z}\setminus\{0\}$. Assume for simplicity $x \geq 0$, we have

$$P_x(X_n = y, \tau_\Delta > n) = \mathbf{1}_{\{xy\geq 0\}}P_x(X_n = y, \tau_0 > n) + P_x(X_n = y, \tau_0 \leq n). \tag{8.11}$$

By the Strong Markov property and the symmetry with respect to 0, the second summand on the righthand side is equal to

$$\frac{1}{2}Q_x(Y_n = |y|, \tau_0^Y \le n) = \frac{1}{2}\left(Q_x(Y_n = |y|) - Q_x(Y_n = |y|, \tau_0^Y > n)\right). \quad (8.12)$$

Combine (8.11)

$$P_x(X_n = y, \tau_\Delta > n) = \frac{1}{2}Q_x(Y_n = |y|) + \frac{1}{2}\mathrm{sgn}(xy)Q_x(Y_n = |y|, \tau_0^Y > n) \quad (8.13)$$

However, the second summand on the righthand side is nonzero only when y and $x - n$ have the same parity.

Thus, we will consider the convergence for $x-n$ even and $x-n$ odd, respectively. In particular, since processes $\mathbf{X}$ and $\mathbf{Y}$ won't be affected by the parity, we sum over all y and use

$$\begin{aligned}
Q_x(\tau_\Delta^Y > n) &\sim e^{-\lambda_0 n}\sqrt{\frac{8}{\pi n^3}}\frac{(1-\alpha\rho)\rho^{-x}}{[(1-\rho^2)(1-\alpha^2)]^2} \\
&\quad \times \begin{pmatrix} 1+\rho^2 \\ 2\rho \end{pmatrix} \cdot \left[\left(\begin{pmatrix} 1+\alpha^2 \\ 2\alpha \end{pmatrix} + (1-\alpha^2)x\begin{pmatrix} 1 \\ \alpha \end{pmatrix}\right) + \left(\begin{pmatrix} 2\alpha \\ 1+\alpha^2 \end{pmatrix} + (1-\alpha^2)x\begin{pmatrix} \alpha \\ 1 \end{pmatrix}\right)\right] \\
&\sim e^{-\lambda_0 n}\sqrt{\frac{8}{\pi n^3}}\frac{(1-\alpha\rho)\rho^{-x}(1+(1-\alpha)x)}{[(1-\rho)(1-\alpha)]^2}
\end{aligned}$$

We have the following two cases:

- If $x - n$ even, using equation (8.13), Lemma 8.2 and Lemma 8.3

$$P_x(X_n = y|\tau_\Delta > n) \sim \frac{1}{2}\frac{Q_x(Y_n = |y|)}{Q_x(\tau_\Delta^Y > n)} + \frac{1}{2}\mathrm{sgn}(xy)\mathbf{1}_{\{y\in 2\mathbb{Z}\}}\frac{Q_x(Y_n = |y|, \tau_0^Y > n)}{Q_x(\tau_\Delta^Y > n)} \quad (8.14)$$

$$\sim \frac{1}{2}(1+\delta_0(y))\nu^{\lambda_0}(|y|) + \frac{1}{2}\mathrm{sgn}(xy)\mathbf{1}_{\{y\in 2\mathbb{Z}\}}h_e(x)\nu^{\lambda_0,0}(|y|) \quad (8.15)$$

- Similarly, if $x - n$ odd,

$$P_x(X_n = y|\tau_\Delta > n) \sim \frac{1}{2}\frac{Q_x(Y_n = |y|)}{Q_x(\tau_\Delta^Y > n)} + \frac{1}{2}\mathrm{sgn}(xy)\mathbf{1}_{\{y\in 2\mathbb{Z}+1\}}\frac{Q_x(Y_n = |y|, \tau_0^Y > n)}{Q_x(\tau_\Delta^Y > n)} \quad (8.16)$$

$$\sim \frac{1}{2}(1+\delta_0(y))\nu^{\lambda_0}(|y|) + \frac{1}{2}\mathrm{sgn}(xy)\mathbf{1}_{\{y\in 2\mathbb{Z}+1\}}h_o(x)\nu^{\lambda_0,0}(|y|) \quad (8.17)$$

where

$$h_e(x) = h_o(x) = \frac{\rho(1-\alpha)^2 x}{(1-\rho\alpha)(1+(1-\alpha)x)}.$$

In addition, observe from (8.10)

$$\frac{1}{2}(\mu_+^{\lambda_0} + \mu_-^{\lambda_0})(y) = \frac{1}{2}(1 + \delta_0(y))\nu^{\lambda_0}(|y|),$$

and

$$\frac{1}{2}(\mu_+^{\lambda_0} - \mu_-^{\lambda_0})(y) = \frac{1}{2}\frac{\rho(1-\alpha)}{1-\rho\alpha}\mathrm{sgn}(y)\nu^{\lambda_0,0}(|y|)$$

Therefore, for $x \in \mathbb{Z}_+, n \in \mathbb{Z}_+$, define

$$h(x) = \frac{(1-\alpha)x}{1+(1-\alpha)x},$$

and

$$\mu^{\lambda_0}(y) = \frac{1}{2}(\mu_+^{\lambda_0} + \mu_-^{\lambda_0})(y),$$

we have

1.

$$\begin{aligned}
&P_x(X_{2n} = y|\tau_\Delta > 2n) \\
&\to \frac{1}{2}(\mu_+^{\lambda_0} + \mu_-^{\lambda_0})(y) + \mathbf{1}_{2\mathbb{Z}}(y-x)\mathrm{sgn}(x)\frac{(1-\alpha)x}{1+(1-\alpha)x}\frac{1}{2}(\mu_+^{\lambda_0} - \mu_-^{\lambda_0})(y) \\
&= \mu^{\lambda_0}(y) + \mathbf{1}_{2\mathbb{Z}}(y-x)\kappa(x)\frac{(\mu_+^{\lambda_0} - \mu_-^{\lambda_0})(y)}{2}
\end{aligned}$$

2.

$$\begin{aligned}
&P_x(X_{2n+1} = y|\tau_\Delta > 2n+1) \\
&\to \frac{1}{2}(\mu_+^{\lambda_0} + \mu_-^{\lambda_0})(y) + \mathbf{1}_{2\mathbb{Z}+1}(y-x)\mathrm{sgn}(x)\frac{(1-\alpha)x}{1+(1-\alpha)x}\frac{1}{2}(\mu_+^{\lambda_0} - \mu_-^{\lambda_0})(y) \\
&= \mu^{\lambda_0}(y) + \mathbf{1}_{2\mathbb{Z}+1}(y-x)\kappa(x)\frac{(\mu_+^{\lambda_0} - \mu_-^{\lambda_0})(y)}{2}
\end{aligned}$$

where

$$\kappa(x) = \mathrm{sgn}(x)h(x)$$

By symmetry with respect to 0 the above results can be extended to $x \in \mathbb{Z}$ with

$$h(x) = \frac{(1-\alpha)|x|}{1+(1-\alpha)|x|},$$

completing the proof. □

8.6 Tail Estimates

In this section, we prove Lemma 8.2 and Lemma 8.3. Recall $\mathbf{Y}$ is a Birth and Death process on $\mathbb{Z}_+ \cup \{\Delta\}$ with Δ being a unique absorbing state and the process $\mathbf{Y}^0$, defined as follows:

$$Y_t^0 = Y_t \mathbf{1}_{\{\tau_0^Y > t\}}.$$

In order to prove Lemma 8.2 and Lemma 8.3, we need an auxiliary result on simple symmetric random walks. Let $\mathbf{S} = (S_n : n \in \mathbb{Z}_+)$ be the simple symmetric random walk on $\mathbb{Z}$. Let $\sigma_0 = \inf\{n \geq 0 : S_n = 0\}$ and write P_x, E_x for the probability and expectation for $\mathbf{S}$ with $S_0 = x$. We omit the subscript and write P and E for the case $x = 0$.

Proof of Lemma 8.2. We assume first $x - y$ and n are both even. For $x, y \geq 1$, the reflection principle gives

$$\begin{aligned} P_x(S_n = y, \sigma_0 > n) &= P_x(S_n = y) - P_x(S_n = -y) \\ &= P(S_n = y - x) - P(S_n = y + x) \end{aligned}$$

Observe that the Q_x probability of each of the paths for $\mathbf{Y}$ on the left-hand side is the probability under the random walk, times the change of measure coefficient, which is $2^n q^{(n-(y-x))/2}(1-q)^{(n+(y-x))/2}$. Therefore we have

$$Q_x(Y_t = y, \tau_0^Y > n) = (2\sqrt{q(1-q)})^n (\sqrt{\frac{1-q}{q}})^{y-x} \left(P(S_t = y - x) - P(S_t = y + x)\right) \tag{8.18}$$

$$= e^{-\lambda_0 n} \rho^{y-x} \left(P(S_n = y - x) - P(S_n = y + x)\right). \tag{8.19}$$

This and the local central limit theorem with estimates for one-dimensional simple symmetric random walk [17, Proposition 2.5.3] give the first claim. It remains to extend it to the case where both $y - x$ and n are odd. In this case, we have

$$\begin{aligned} P(S_n = y - x) - P(S_n = y + x) &= \frac{1}{2}\left(P(S_{n-1} = y - x - 1) - P(S_{n-1} = y + 1 + x)\right) \\ &\quad + \frac{1}{2}\left(P(S_{n-1} = y - x + 1) - P(S_{n-1} = y + x - 1)\right) \\ &= \sqrt{\frac{8}{\pi(n-1)^3}} \left(xy + \frac{y}{2}((x+1)\eta_1 + (x-1)\eta_2)\right) \end{aligned}$$

As a result, Lemma 8.2-claim 1 holds whenever $y - x$ and n have the same parity.

For the second claim, when summing over y using (8.19), we split the summation according to whether $y \leq n^{1/3}\delta(n)$ or $y > n^{1/3}\delta(n)$,

$$\begin{aligned} Q_x(\tau_0^Y > n) &= e^{-\lambda_0 n}\Big[\sum_{y=1}^{n^{1/3}\delta(n)} + \sum_{y > n^{1/3}\delta(n)} \rho^{y-x}(P(S_t = y - x) - P(S_t = y + x))\Big] \\ &= I_1 + I_2 \end{aligned}$$

Where

$$I_1 \sim e^{-\lambda_0 n}\sqrt{\frac{8}{\pi n^3}}x\rho^{-x}\sum_{y\in\mathbb{N},y-x-n\in 2\mathbb{Z}}^{n^{1/3}\ \delta(n)} y\rho^y.$$

Thus, we need to consider the following two cases:

- $x-n\in 2\mathbb{Z}$. Then we need to sum over $y\in 2\mathbb{N}$:

$$\sum_{y\in 2\mathbb{N},y\le n^{1/3}\delta(n)} y\rho^y \sim 2\sum_{z\in\mathbb{N}} z(\rho^2)^z \sim \frac{2\rho^2}{(1-\rho^2)^2}. \tag{8.20}$$

- $x-n\in 2\mathbb{Z}-1$. Then the summation is over odd y and is therefore asymptotically equivalent to

$$\sum_{y=1}^{\infty} y\rho^y - \sum_{y\in 2\mathbb{N}} y\rho^y \sim \frac{\rho}{(1-\rho)^2} - \frac{\rho^2}{(1-\rho^2)^2} \tag{8.21}$$

$$= \frac{\rho}{(1-\rho)^2}\left(1-\frac{2\rho}{(1+\rho)^2}\right) \tag{8.22}$$

$$= \frac{\rho(1+\rho^2)}{(1-\rho^2)^2}. \tag{8.23}$$

This gives the asymptotic for I_1. As $0\le I_2\le C(n)x\rho^{-x}\dfrac{n^{1/3}\delta(n)+1}{(1-\rho)^2}\rho^{n^{1/3}\delta(n)+1} = o(I_1)$, the proof is now complete. □

Next, we give the proof of Lemma 8.3. We will break it into two pieces and show the second claim first.

Proof of Lemma 8.3-claim 2. Observe that the distribution of τ_Δ^Y under Q_x is the same as the distribution of τ_0^Y under $Q_{x+\mathrm{Geom}(\delta/q)}$. In particular, this gives

$$Q_x(\tau_\Delta^Y > n) = \sum_{k=1}^{\infty}\frac{\delta}{q}(1-\frac{\delta}{q})^{k-1}Q_{x+k}(\tau_0^Y > n).$$

We will write the sum on the right-hand side as $J_1+J_2+J_3$, where J_1 is the summation over $k=1$ to $k=n^{1/3}\gamma(n)$. J_2 is over $n^{1/3}\gamma(n)<k<n$ and J_3 is over $k>n$. Clearly, $J_3\le(1-\delta/q)^{n+1}$, but since $\delta>\delta_{cr}$, $1-\delta/q<e^{-\lambda_0}$ and so J_3 decays to zero at a geometric rate faster than $e^{-\lambda_0}$. Also since

$$\begin{aligned} Q_x(\tau_0^Y > n) &= e^{-\lambda_0 n}\left(E[\rho^{S_n}, S_n > -x] - \rho^{-2x}E[\rho^{S_n}, S_n > x]\right) \\ &= e^{-\lambda_0 n}\left(E[\rho^{S_n}, -x < S_n \le x] + (1-\rho^{-2x})E[\rho^{S_n}, S_n > x]\right). \end{aligned}$$

Then for J_2 we will use the upper bound $Q_{x+k}(\tau_0^Y > n)\le e^{-\lambda_0 n}E[\rho^{S_n}, S_n > -x-k]\le e^{-\lambda_0 n}\rho^{-x-k+1}(1-\rho)^{-1}$. In particular,

$$J_2 \le e^{-\lambda_0 n}\frac{\delta}{q}\rho^{-x}(1-\rho)^{-1}\sum_k (1-\delta/q)^{k-1}\rho^{-k+1}.$$

where $(1-\delta/q)/\rho = \alpha < 1$, so we have

$$J_2 \le e^{-\lambda_0 n}\frac{\delta}{q}\alpha^{n^{1/3}\gamma(n)}\rho^{-x}(1-\rho)^{-1} = o(e^{-\lambda_0 t}). \tag{8.24}$$

Finally, we can turn to J_1. We have

$$J_1 = J_{1,1} + J_{1,2},$$

where

$$J_{1,j} = \frac{\delta}{q}\sum_{k=1}^{n^{1/3}\gamma(n)}(1-\delta/q)^{k-1}I_j(n, x+k).$$

To simplify the analysis, we will use $(1-\delta/q)\rho^{-1} = \alpha$, $\dfrac{\delta}{q} = 1-\alpha\rho$. Thus

$$\begin{aligned}
J_{1,1} &= \frac{\delta}{q}\sum_{k=1}^{n^{1/3}\gamma(n)}\sum_{y=1}^{2n^{1/3}\gamma(n)} e^{-\lambda_0 t}(1-\delta/q)^{k-1}\rho^{y-x-k}(P(S_t = y-x-k) - P(S_t = y+x+k)) \\
&\sim \frac{\delta}{q}e^{-\lambda_0 n}\sqrt{\frac{8}{\pi n^3}}\rho^{-x}\left(\sum_{k=1}^{n^{1/3}\gamma(n)}(1-\delta/q)^{k-1}\rho^{-k}(x+k)\right)\left(\sum_{y=1}^{2t^{1/3}\gamma(n)}\rho^y y\right) \\
&\sim \frac{\delta}{q}e^{-\lambda_0 n}\sqrt{\frac{8}{\pi n^3}}\rho^{-x-1}\alpha^{-x}\left(\sum_{k=1}^{n^{1/3}\gamma(n)}\alpha^{x+k-1}(x+k)\right)\left(\sum_{y=1}^{2t^{1/3}\gamma(n)}\rho^y y\right).
\end{aligned}$$

Since $y-(x+k)-n$ must be even, we need to consider the following two cases for the two summations above:

- $x+n \in 2\mathbb{Z}$. Then $y-k \in 2\mathbb{Z}$:

$$\begin{aligned}
J_{1,1} \sim \frac{\delta}{q}e^{-\lambda_0 n}\sqrt{\frac{8}{\pi n^3}}\frac{\rho^{-x-1}}{\alpha^x}\Bigg[&\left(\sum_{k\in 2\mathbb{N}, k\le n^{1/3}\gamma(n)}\alpha^{x+k-1}(x+k)\right)\left(\sum_{y\in 2\mathbb{N}, y\le 2n^{1/3}\gamma(n)}\rho^y y\right) \\
&+\left(\sum_{k\in 2\mathbb{N}-1, k\le n^{1/3}\gamma(n)}\alpha^{x+k-1}(x+k)\right)\left(\sum_{y\in 2\mathbb{N}-1, y\le 2t^{1/3}\gamma(n)}\rho^y y\right)\Bigg]
\end{aligned}$$

  Thus, using (8.20) and (8.23)

$$J_{1,1} \quad \sim e^{-\lambda_0 n}\sqrt{\frac{8}{\pi n^3}}\frac{(1-\alpha\rho)\rho^{-x}}{[(1-\rho^2)(1-\alpha^2)]^2}\begin{pmatrix}1+\rho^2\\ 2\rho\end{pmatrix}\cdot\left[\begin{pmatrix}1+\alpha^2\\ 2\alpha\end{pmatrix} + (1-\alpha^2)x\begin{pmatrix}1\\ \alpha\end{pmatrix}\right]$$

- $x+n \in 2\mathbb{Z}-1$. Then $y-k \in 2\mathbb{Z}-1$:

$$J_{1,1} \sim \frac{\delta}{q} e^{-\lambda_0 n} \sqrt{\frac{8}{\pi n^3}} \frac{\rho^{-x-1}}{\alpha^x} \left[\left(\sum_{k \in 2\mathbb{N}, k \le n^{1/3}\gamma(n)} \alpha^{x+k-1}(x+k) \right) \left(\sum_{y \in 2\mathbb{N}-1, y \le 2t^{1/3}\gamma(n)} \rho^y y \right) \right.$$
$$\left. + \left(\sum_{k \in 2\mathbb{N}-1, k \le n^{1/3}\gamma(n)} \alpha^{x+k-1}(x+k) \right) \left(\sum_{y \in 2\mathbb{N}, y \le 2t^{1/3}\gamma(n)} \rho^y y \right) \right]$$

Thus, (8.20) and (8.23) imply

$$J_{1,1} \quad \sim e^{-\lambda_0 n} \sqrt{\frac{8}{\pi n^3}} \frac{(1-\alpha\rho)\rho^{-x}}{[(1-\rho^2)(1-\alpha^2)]^2} \begin{pmatrix} 1+\rho^2 \\ 2\rho \end{pmatrix} \cdot \left[\begin{pmatrix} 2\alpha \\ 1+\alpha^2 \end{pmatrix} + (1-\alpha^2)x \begin{pmatrix} \alpha \\ 1 \end{pmatrix} \right]$$

Next, notice that

$$J_{1,2} \le \frac{\delta}{q} e^{-\lambda_0 n} \sum_{k=1}^{n^{1/3}\gamma(n)} (1-\delta/q)^{k-1} \sum_{y > 2n^{1/3}\delta(n)} \left[\rho^{y-x-k} (P(S_t = y-x-k) - P(S_t = y+x+k)) \right]$$
$$\le \frac{\delta}{q} e^{-\lambda_0 n} \sqrt{\frac{8}{\pi n^3}} \rho^{-x-1} \alpha^{-x} \left(\sum_{k=1}^{n^{1/3}\gamma(n)} \alpha^{x+k-1}(x+k) \right) \left(\sum_{y > 2n^{1/3}\delta(n)} \rho^y y \right)$$
$$\le e^{-\lambda_0 n} \sqrt{\frac{8}{\pi n^3}} \frac{(1-\alpha\rho)\rho^{-x}}{[(1-\rho)(1-\alpha)]^2} \left((1-\alpha)x + 1 \right) (2n^{1/3}\delta(n) + 1) \rho^{2n^{1/3}\delta(n)} = o(J_{1,1}).$$

This and (8.24) then give $J_1 + J_2 \sim J_{1,1}$, completing the proof.

□

We now compute $Q_x(Y_n = y, \tau_\Delta^Y > n)$. We begin by finding the generating function of expression, $\phi_x(\beta, y)$, defined as

$$\phi_x(\beta, y) = \sum_{n=0}^{\infty} \beta^n Q_x(Y_t = y, \tau_\Delta^Y > n).$$

By conditioning on the transition from Y_{n-1} to Y_t, whenever $y > 0$, we have that for all $n > 0$

$$Q_x(Y_n = y, \tau_\Delta^Y > n) = Q_x(Y_{n-1} = y-1)(1-q) + Q_x(Y_{n-1} = y+1)q$$

Therefore

$$\phi_x(\beta, y) = Q_x(Y_0 = y) + (1-q)\beta \sum_{n=1}^{\infty} \beta^{n-1} Q_x(Y_{n-1} = y-1) + q\beta \sum_{n=1}^{\infty} \beta^{n-1} Q_x(Y_{n-1} = y+1)$$
$$= \delta_x(y) + (1-q)\beta\phi_x(\beta, y-1) + q\beta\phi_x(\beta, y+1). \tag{8.25}$$

Similarly when $y = 0$, for $n > 0$ we have

$$Q_x(Y_t = 0) = Q_x(Y_{n-1} = 0)r + Q_x(Y_{n-1} = 1)q,$$

so that

$$\phi_x(\beta, 0) = \delta_x(0) + r\beta\phi_x(\beta, 0) + q\beta\phi_x(\beta, 1). \tag{8.26}$$

The calculation of difference equations (Appendix 8.6) gives

$$\begin{aligned}\phi_x(\beta, y) = (q\beta)^{-y} &\left[\frac{\eta^{y+1} - (1-\eta)^{y+1}}{2\eta - 1}\phi_x(\beta, 0) - \frac{\eta^y - (1-\eta)^y}{2\eta - 1}\left(r\beta\phi_x(\beta, 0) + \delta_{x,0}\right)\right] \\ &+ \mathbf{1}_{\{y>x\}}(q\beta)^x \frac{\eta^{y-x} - (1-\eta)^{y-x}}{2\eta - 1}\end{aligned}$$

Where $\eta = \gamma^{-1}\beta q$ and $1 - \eta = \gamma\beta(1-q)$ for some $\gamma > 0$. Using this, we can complete the proof of Lemma 8.3.

Proof of Lemma 8.3-Claim 1. Consider

$$\bar{\phi}_x(\beta, y_0) = \sum_{y \geq y_0} \phi_x(\beta, y) = I_1 + I_2 + I_3.$$

where

$$\begin{aligned}I_1 &= \frac{\phi_x(\beta, 0)(q\beta)^{-y_0}}{2\eta - 1} \sum_{y \geq y_0} (q\beta)^{-(y-y_0)} \left(\eta^y(\eta - r\beta) - (1-\eta)^y((1-\eta) - r\beta)\right) \\ &= \frac{\phi_x(\beta, 0)(q\beta)^{-y_0+1}}{2\eta - 1}\left(\frac{\eta^{y_0}(\eta - r\beta)}{\beta q - \eta} - \frac{(1-\eta)^{y_0}((1-\eta) - r\beta)}{\beta q - (1-\eta)}\right)\end{aligned}$$

We expand according to $\eta^y = 2^{-y}(1-\epsilon)^y$ and $(1-\eta)^y = 2^{-y}(1+\epsilon)^y$ for the general y, only using the first two terms (coefficients of ϵ^0 and ϵ^1). All other terms will lead to contributions of smaller orders. Hence We have

$$\begin{aligned}\bar{\phi}_x(\beta, y) &\sim \frac{\phi_x(\beta, 0)}{2\eta - 1}(\beta q)^{-y+1}2^{-y}\left(\frac{(1-\epsilon)^y(\eta - r\beta)}{\beta q - \eta} - \frac{(1+\epsilon)^y((1-\eta) - r\beta)}{\beta q - (1-\eta)}\right) \\ &= (2\beta q)^{-y}\bar{\phi}_x(\beta, 0) + (2\beta q)^{-y} y\phi_x(\beta, 0)(\beta q)\left(\frac{\eta - r\beta}{\beta q - \eta} + \frac{(1-\eta) - r\beta}{\beta q - (1-\eta)}\right) \\ &= (2\beta q)^{-y}\bar{\phi}_x(\beta, 0)\left(1 + y\frac{q + r - 2\beta q(1 - q + r)}{q - r}\right).\end{aligned}$$

Where

$$\bar{\phi}_x(\beta, 0) = \sum_{y \geq 0} \phi_x(\beta, y) = \phi_x(\beta, 0)\beta q \times \frac{\beta(q-r)}{(\beta q - \eta)(\beta q - (1-\eta))}.$$

Therefore, taking $\beta = e^{\lambda_0}$ we have

$$Q_x(Y_n \geq y, \tau_\Delta^Y > n) \sim \rho^y\left(1 + y\frac{q + r - \rho^{-1}(1 - q + r)}{q - r}\right)Q_x(\tau_\Delta^Y > n).$$

Observe that

$$\frac{q+r-\rho^{-1}(1-q+r)}{q-r} = \frac{(\sqrt{q}-\sqrt{1-q})(1-\alpha)}{\sqrt{q}-\sqrt{1-q}\alpha} = \frac{1-\rho}{1-\rho\alpha}(1-\alpha),$$

so,

$$Q_x(Y_n \geq y, \tau_\Delta^Y > n) \sim \rho^y(1+\underbrace{y\frac{1-\rho}{1-\rho\alpha}(1-\alpha)}_{=C})Q_x(\tau_\Delta^Y > n).$$

Hence,

$$\begin{aligned} Q_x(Y_n = y, \tau_\Delta^Y > n) &= Q_x(Y_t \geq y, \tau_\Delta^Y > n) - Q_x(Y_t \geq y+1, \tau_\Delta^Y > n) \\ &\sim \rho^y\left((1+Cy) - \rho(1+C(y+1))\right) Q_x(\tau_\Delta^Y > n) \\ &= \frac{(1-\rho)^2}{1-\rho\alpha}\rho^y\left(1+(1-\alpha)y\right) Q_x(\tau_\Delta^Y > n). \end{aligned}$$

□

9 Appendix: Calculations for Section 8.6

To solve difference equations (8.25) and (8.26), we fix $\beta < 1$. We will find $\gamma > 0$ such that

$$\underbrace{\gamma\beta(1-q)}_{=1-\eta} + \underbrace{\gamma^{-1}\beta q}_{=\eta} = 1.$$

Setting $H_x(\beta, y) = \gamma^y \phi_x(\beta, y)$, (8.25) and (8.26) are equivalent, respectively, to

$$H_x(\beta, y) = \gamma^y \delta_{x,y} + (1-\eta)H_x(\beta, y-1) + \eta H_x(\beta, y+1), \ y \geq 1. \tag{9.1}$$

$$H_x(\beta, 0) = \delta_{x,0} + r\beta H_x(\beta, 0) + \eta H_x(\beta, 1) \tag{9.2}$$

We first solve the system for the case $y+1 \leq x$. In this case, (9.1) gives

$$H_x(\beta, y+1) - H_x(\beta, y) = \frac{1-\eta}{\eta}\left(H_x(\beta, y) - H_x(\beta, y-1)\right) \tag{9.3}$$

and

$$H_x(\beta, 1) - H_x(\beta, 0) = \underbrace{\frac{(1-\eta-r\beta)H_x(\beta, 0) - \delta_{x,0}}{\eta}}_{I_x(\beta)}.$$

Hence we have

$$H_x(\beta, j+1) - H_x(\beta, j) = \left(\frac{1-\eta}{\eta}\right)^j \left(H_x(\beta, 1) - H_x(\beta, 0)\right), j \geq 2$$

Summing from $j = 1$ to $y - 1$, we then obtain

$$H_x(\beta, y) - H_x(\beta, 1) = \sum_{j=1}^{y-1} \left(\frac{1-\eta}{\eta}\right)^j I_x(\beta).$$

Hence we have

$$H_x(\beta, y) = \frac{1-\eta}{\eta} \frac{1 - (\frac{1-\eta}{\eta})^{y-1}}{1 - \frac{1-\eta}{\eta}} I_x(\beta) + \underbrace{H_x(\beta, 1)}_{=I_x(\beta)+H_x(\beta,0)}, \ y \leq x.$$

Next, when $y = x$, we have the equation

$$\begin{aligned} H_x(\beta, x+1) - H_x(\beta, x) &= \frac{(1-\eta)(H_x(\beta, x) - H_x(\beta, x-1)) - \gamma^x}{\eta} \\ &= (\frac{1-\eta}{\eta})^x I_x(\beta) - \frac{\gamma^x}{\eta}, \end{aligned}$$

and since for $y > x$ we can also use (9.3), we have

$$H_x(\beta, y+1) - H_x(\beta, y) = (\frac{1-\eta}{\eta})^y I_x(\beta) - (\frac{1-\eta}{\eta})^{y-x} \frac{\gamma^x}{\eta}, \ y > x$$

Which gives the following formula

$$H_x(\beta, y) - H_x(\beta, x) = \sum_{j=x}^{y-1} \left((\frac{1-\eta}{\eta})^j I_x(\beta) - (\frac{1-\eta}{\eta})^{j-x} \frac{\gamma^x}{\eta} \right), \ y \geq x+1$$

Altogether,

$$\begin{aligned} H_x(\beta, y) &= \sum_{j=1}^{y-1} (\frac{1-\eta}{\eta})^j I_x(\beta) - \frac{\gamma^x}{\eta} \sum_{j=x}^{y-1} (\frac{1-\eta}{\eta})^{j-x} + H_x(\beta, 1) \\ &= \sum_{j=0}^{y-1} (\frac{1-\eta}{\eta})^j I_x(\beta) - \frac{\gamma^x}{\eta} \sum_{j=x}^{y-1} (\frac{1-\eta}{\eta})^{j-x} + H_x(\beta, 0). \end{aligned}$$

To get $\phi_x(\beta, y)$, multiply by γ^{-y} and let $\omega = \dfrac{1-\eta}{\eta}$, we obtain

$$\phi_x(\beta, y) = \gamma^{-y} \frac{1-\omega^y}{1-\omega} I_x(\beta) - \frac{\gamma^{x-y}}{\eta} \frac{1-\omega^{y-x}}{1-\omega} + \gamma^{-y} H_x(\beta, 0).$$

Summing over all y, we have

$$\frac{\left(\frac{1}{1-\gamma^{-1}} - \frac{1}{1-\omega/\gamma}\right)}{1-\omega} I_x(\beta) + \frac{1}{1-\gamma^{-1}} H_x(\beta, 0) - \frac{\left(\frac{1}{1-\gamma^{-1}} - \frac{1}{1-\omega/\gamma}\right)}{\eta(1-\omega)}.$$

Using $\gamma^{-1}\beta q = \eta$ we get

$$\begin{aligned}\phi_x(\beta, \mathbf{1}) = (\beta q)^2 \frac{1}{(\beta q - \eta)[\beta q - (1-\eta)]} \phi_x(\beta, 0) + \beta q \frac{1}{(\beta q - \eta)[\beta q - (1-\eta)]} (-r\beta)\phi_x(\beta, 0) \\ - \beta q \frac{1}{(\beta q - \eta)[\beta q - (1-\eta)]} (1 + \delta_{x,0})\end{aligned}$$

To get back to ϕ_x, we need to multiply both sides by γ^{-y} and use the identity (9.3). This gives

$$\phi_x(\beta, y) = \begin{cases} (q\beta)^{-y} \left[\frac{\eta^{y+1} - (1-\eta)^{y+1}}{2\eta - 1} \phi_x(\beta, 0) - \frac{\eta^y - (1-\eta)^y}{2\eta - 1} \left(r\beta\phi_x(\beta, 0) + \delta_{x,0}\right)\right], \ y \leq x \\ (q\beta)^{-y} \left[\frac{\eta^{y+1} - (1-\eta)^{y+1}}{2\eta - 1} \phi_x(\beta, 0) - \frac{\eta^y - (1-\eta)^y}{2\eta - 1} \left(r\beta\phi_x(\beta, 0) + \delta_{x,0}\right) + (\beta q)^x \frac{\eta^{y-x} - (1-\eta)^{y-x}}{2\eta - 1}\right], \ y > x \end{cases}$$

References

[1] William J Anderson. *Continuous-time Markov chains: An applications-oriented approach*. New York, NY: Springer, 2012. DOI: https://doi.org/10.1007/978-1-4612-3038-0.

[2] Krishna B. Athreya and Samuel Karlin. "On Branching Processes with Random Environments: I: Extinction Probabilities". In: *The Annals of Mathematical Statistics* 42 (5 1971). ISSN: 0003-4851. DOI: 10.1214/aoms/1177693150.

[3] M. S. Bartlett. "On Theoretical Models for Competitive and Predatory Biological Systems". In: *Biometrika* 44.1/2 (1957), pp. 27–42. ISSN: 00063444. URL: http://www.jstor.org/stable/2333238 (visited on 07/12/2023).

[4] M. G. Bulmer and M. S. Bartlett. "Stochastic Population Models in Ecology and Epidemiology." In: *Applied Statistics* 10 (3 1961). ISSN: 00359254. DOI: 10.2307/2985209.

[5] Nicolas Champagnat and Denis Villemonais. *Exponential convergence to quasi-stationary distribution and Q-process*. 2014. arXiv: 1404.1349 [math.PR].

[6] Nicolas Champagnat and Denis Villemonais. "General criteria for the study of quasi-stationarity". In: *Electronic Journal of Probability* 28.none (2023), pp. 1–84. DOI: 10.1214/22-EJP880. URL: https://doi.org/10.1214/22-EJP880.

[7] Bertrand Cloez and Marie-Noémie Thai. *Quantitative results for the Fleming-Viot Particle system and quasi-stationary distributions in discrete space*. 2014. arXiv: 1312.2444 [math.PR].

[8] Vere-Jones David. "Ergodic properties of nonnegative matrices. I". In: *Pacific Journal of Mathematics* 22.2 (Aug. 1967), pp. 361–386. ISSN: 0030-8730. DOI: 10.2140/pjm.1967.22.361. URL: https://cir.nii.ac.jp/crid/1363951793326005248.

[9] P. A. Ferrari, H. Kesten, and S. Martínez. "R-positivity, quasi-stationary distributions and ratio limit theorems for a class of probabilistic automata". In: *The Annals of Applied Probability* 6.2 (1996), pp. 577–616. DOI: 10.1214/aoap/1034968146. URL: https://doi.org/10.1214/aoap/1034968146.

[10] P. A. Ferrari et al. "Existence of Quasi-Stationary Distributions. A Renewal Dynamical Approach". In: *The Annals of Probability* 23.2 (1995), pp. 501–521. DOI: 10.1214/aop/1176988277. URL: https://doi.org/10.1214/aop/1176988277.

[11] James Fill. "On Hitting Times and Fastest Strong Stationary Times for Skip-Free and More General Chains". In: *Journal of Theoretical Probability* 22 (Aug. 2007). DOI: 10.1007/s10959-009-0233-7.

[12] C. M. Fortuin, J. Ginibre, and P. W. Kasteleyn. "Correlation inequalities on some partially ordered sets". In: *Communications in Mathematical Physics* 22.2 (1971), pp. 89–103.

[13] Sophie Hautphenne and Stefano Massei. "A Low-Rank Technique for Computing the Quasi-Stationary Distribution of Subcritical Galton–Watson Processes". In: *SIAM Journal on Matrix Analysis and Applications* 41.1 (2020), pp. 29–57. DOI: 10.1137/19M1241647. eprint: https://doi.org/10.1137/19M1241647. URL: https://doi.org/10.1137/19M1241647.

[14] E. Hille and R.S. Phillips. *Functional Analysis and Semi-groups*. American mathematical society colloquium publications. American Mathematical Society, 1974. URL: https://books.google.com/books?id=peTIoQEACAAJ.

[15] Tseng Fung Ho, Chi Chung Lin, and Chih Ling Lin. "Using fuzzy sets and Markov chain method to carry out inventory strategies with different recovery levels". In: *Symmetry* 12 (8 2020). ISSN: 20738994. DOI: 10.3390/SYM12081226.

[16] Anthony W. Knapp John G. Kemeny J. Laurie Snell. *Denumerable Markov chains / John G. Kemeny, J. Laurie Snell, Anthony W. Knapp*. English. 2d. ed. Springer-Verlag New York, 1976, xii, 484 p. ISBN: 0387901779.

[17] Gregory F. Lawler and Vlada Limic. *Random Walk: A Modern Introduction*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2010. DOI: 10.1017/CBO9780511750854.

[18] Servet Martínez. "Quasi-Stationary Distributions for Birth-Death Chains. Convergence Radii and Yaglom Limit". In: *Cellular Automata and Cooperative Systems*. Ed. by Nino Boccara et al. Dordrecht: Springer Netherlands, 1993, pp. 491–505. ISBN: 978-94-011-1691-6. DOI: 10.1007/978-94-011-1691-6_39. URL: https://doi.org/10.1007/978-94-011-1691-6_39.

[19] Servet Martínez, Jaime San Martín, and Denis Villemonais. "Existence and uniqueness of a quasistationary distribution for markov processes with fast return from infinity". In: *Journal of Applied Probability* 51 (3 2014). ISSN: 00219002. DOI: 10.1239/jap/1409932672.

[20] Servet Martínez and Maria Eulália Vares. "A Markov Chain Associated with the Minimal Quasi-Stationary Distribution of Birth-Death Chains". In: *Journal of Applied Probability* 32.1 (1995), pp. 25–38. ISSN: 00219002. URL: http://www.jstor.org/stable/3214918 (visited on 06/26/2023).

[21] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*. eng. Applied mathematical sciences ; v. 44. New York: Springer-Verlag, 1983. ISBN: 0387908455.

[22] Jaime San Martín Pierre Collet Servet Martínez. "Quasi-Stationary Distributions: Markov Chains, Diffusions and Dynamical Systems". In: (2013). DOI: https://doi.org/10.1007/978-3-642-33131-2.

[23] Stanley A. Sawyer. "Martin boundaries and random walks". In: (1997). DOI: 10.1090/conm/206/02685.

[24] E. Seneta and D. Vere-Jones. "On quasi-stationary distributions in discrete-time Markov chains with a denumerable infinity of states". In: *Journal of Applied Probability* 3 (2 1966). ISSN: 0021-9002. DOI: 10.2307/3212128.

[25] Oliver Tough. *L^∞-convergence to a quasi-stationary distribution*. 2022. arXiv: 2210.13581 [math.PR].

[26] Erik A. Van Doorn. "Quasi-stationary distributions and convergence to quasi-stationarity of birth-death processes". In: *Advances in Applied Probability* 23.4 (1991), pp. 683–700. DOI: 10.2307/1427670.

[27] Erik A. van Doorn and Philip K. Pollett. "Quasi-stationary distributions for discrete-state models". In: *European Journal of Operational Research* 230.1 (2013), pp. 1–14. ISSN: 0377-2217. DOI: https://doi.org/10.1016/j.ejor.2013.01.032. URL: https://www.sciencedirect.com/science/article/pii/S0377221713000799.

[28] D. Vere-Jones. "Some limit theorems for evanescent processes". In: *Australian Journal of Statistics* 11 (2 1969). ISSN: 1467842X. DOI: 10.1111/j.1467-842X.1969.tb00300.x.

[29] Wolfgang Woess. "Denumerable Markov Chains". In: (2009). DOI: 10.4171/071.

[30] Sewall Wright. "EVOLUTION IN MENDELIAN POPULATIONS". In: *Genetics* 16.2 (Mar. 1931), pp. 97–159. ISSN: 1943-2631. DOI: 10.1093/genetics/16.2.97. eprint: https://academic.oup.com/genetics/article-pdf/16/2/97/35081059/genetics0097.pdf. URL: https://doi.org/10.1093/genetics/16.2.97.

[31] A. M. Yaglom. "Certain limit theorems of the theory of branching random processes". In: *Dokl. Akad. Nauk SSSR, n. Ser.* 56 (1947), pp. 795–798. ISSN: 0002-3264.