Quasi-Stationary Distributions for the Voter and Invasion Dynamics on Complete Bipartite Graphs

Symposium on Stochastic Hybrid Systems and Applications

Iddo Ben-Ari (University of Connecticut), based on work with Hugo Panzo (Technion) and Undergraduate Researchers from the UConn Markov Chains REU[1], '19,'20,'21

UConn, 2021-11-12

[1] For applications, lookup "MCREU22" on Mathprograms.org early in January.

Outline

Quasistationary Distributions

Assumptions

$\mathbf{Y} = (Y_t : t \in \mathbb{Z}_+)$ A discrete-time MC on a finite state space $\Omega \cup \Delta$ with TF p and

- Δ is **absorbing**: $P_\delta(Y_1 \in \Delta) = 1, \ \delta \in \Delta$;
- Δ accessible from Ω.
- The restriction of p to $\Omega \times \Omega$, $p|\Omega$, is irreducible.

Define the **absorption time**

$$\tau = \inf\{t \in \mathbb{Z}_+ : Y_t = \Delta\}.$$

From assumptions, τ has geometric tails.

All Stationary distributions supported on Δ, so the next best thing may be

Definition 1 (QSD)

A probability distribution ν on Ω is a **quasistationary distribution** *(QSD) if*

$$P_\nu(Y_t \in \cdot \,|\tau > t) = \nu, \ \ t \in \mathbb{Z}_+.$$

Note

- Everything has to end. How would it look if it lasted very long?

General Results

Proposition 1 (QSD Characterization)

A probability vector ν on Ω is a QSD if and only if it is a left **Perron Eigenvector** *for $p|_\Omega$. That is,*

$$\nu p|_\Omega = \lambda \nu \tag{1}$$

for some (any) λ. In this case λ is the Perron eigenvalue/spectral radius for $p|_\Omega$

Note

When Ω is infinite (still irreducible): Existence and Uniqueness are not guaranteed (all possibilities can be realized through B&D on $\mathbb{Z}_+$).

Probability notation

For every initial distribution μ on Ω and $t \in \mathbb{Z}_+$,

$$P_\mu(Y_t = \cdot, \tau > t) = \mu p|_\Omega^t(\cdot)$$

Thus with ν the QSD

$$P_\nu(\tau > t) = \nu(p|_\Omega)^t \mathbf{1}_\Omega = \lambda^t.$$

We have

Corollary 1

1. *The distribution of τ under P_ν is Geom$(1-\lambda)$.*
2. $\lambda = \lim_{t\to\infty} (P_x(\tau > t))^{1/t} = \lim_{t\to\infty} (\max_x P_x(\tau > t))^{1/t} = \max_x (\lim_{t\to\infty} P_x(\tau > t)^{1/t})$.

Convergence Theorem

In analogy to stationary distributions we have:

Theorem 1 (Convergence to QSD)

If, in addition, $p|_\Omega$ is aperiodic, then for any initial distribution μ on Ω

$$\lim_{t\to\infty} P_\mu(Y_t \in \cdot \,|\tau > t) = \nu.$$

Note

- From linear algebra,

$$\|P_\mu(Y_t \in \cdot \,|\tau > t) - \nu\|_{TV} = O\left(\left(\frac{|\lambda_2|}{\lambda}\right)^t\right),$$

  where λ_2 is a subdominant eigenvalue for $p|_\Omega$. This may decay faster than $P(\tau > t)$, and so QSD may be observed early in the evolution.
- Nevertheless, in principle sampling QSDs through simulations is a challenge as they emerge as limits under geometrically vanishing events.

Example: RW on the Cycle

Example 2 (QSD for RW on cycle)

Consider simple symmetric RW on the N-cycle $\mathbb{Z}_N = \{0, \ldots, N-1\}$, with 0 as absorbing state. The matrix $p|_\Omega$ is as in the Figure below.

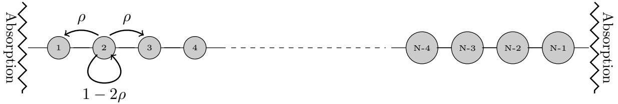

Figure: RW absorbed at 0

- The QSD is a probability ν on $\{1, \ldots, N-1\}$ (extend it to $\mathbb{Z}_N$ by setting $\nu(0) = 0$) satisfying (1):

$$\nu(x-1)p(x-1,x) \quad + \nu(x)p(x,x) \qquad\qquad + \nu(x+1)p(x+1,x) \quad = \lambda\nu(x).$$

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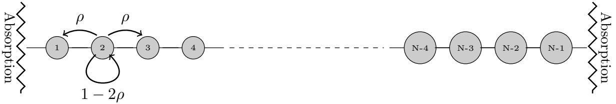

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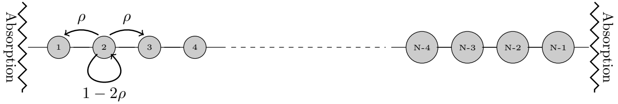

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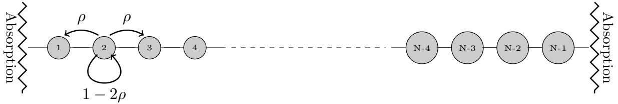

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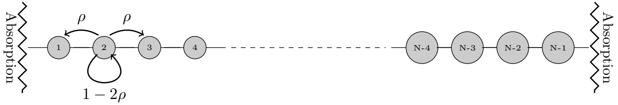

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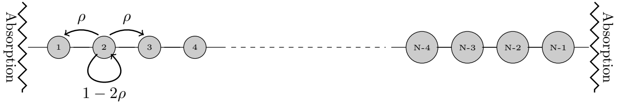

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Equivalently,

$$\frac{1}{2}\left(\nu(x-1) + \nu(x+1)\right) = \frac{2}{\rho}(\lambda - (1-2\rho))\nu(x)$$

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- The solution is then

$$\begin{cases} \nu(x) = C_N \sin(\frac{x}{N}\pi) & (C_N = \tan\frac{\pi}{2N}); \\ \lambda = \frac{\rho}{2}\cos\frac{\pi}{N} + (1-2\rho) \end{cases} \tag{3}$$

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Observations

- Density higher away from absorbing state.
- Continuum limit of model and respective QSD: BM on $[0, 1]$ absorbed at endpoints.

Discrete-time Voter and Invasion

Models describing evolution of "opinions" on a graph in different "cultures".

Setup

- $G = (V, E)$ finite, connected graph.
- State space: **assignments of opinions**, functions $\eta : V \to \mathcal{O}$, where $\mathcal{O}$ is the set of "opinions". We often take either $\mathcal{O} = \{0 = \text{"}no\text{"}, 1 = \text{"}yes\text{"}\}$ or $\mathcal{O} = V$. For a state η, $\eta(v)$ is the opinion of v.

Time Evolution

At time $t \in \mathbb{Z}_+$, opinions are η_t. We sample

- Uniformly a vertex u and a uniformly a neighbor v independently of the past;
- Assign opinions as follows:

Voter

$$\eta_{t+1}(x) = \begin{cases} \eta_t(v) & x = u \\ \eta_t(x) & \text{otherwise} \end{cases}$$

Invasion

$$\eta_{t+1}(x) = \begin{cases} \eta_t(u) & x = v \\ \eta_t(x) & \text{otherwise} \end{cases}$$

Note

- The constant assignments (e.g. all "no"), also known as **consensus** states. are the absorbing set Δ.
- Representing two extreme "cultures": Voter dominated by obedience (?), and Invasion dominated by, well, desire to dominate (?).

(Sightly) More general evolution

In both models, the dynamics is determined only through IID sampling of ordered edges that easily generalizes to

Definition 2 (General Evolution)

Let ρ be a probability measure on the set of order pairs $\{(v,u) : \{v,u\} \in E, u \neq v\}$, with full support. Define an evolution on assignments of opinions as follows:

- *At time $t \in \mathbb{Z}_+$ sample (v,u) according to ρ, independently of the past.*
- *At time $t+1$ assign the opinion of v to u and keep all other opinions unchanged:*

$$\eta_{t+1}(x) = \begin{cases} \eta_t(v) & x = u \\ \eta_t(x) & otherwise \end{cases}$$

Example 3

	Voter	Invasion
$\rho(v,u) =$	$\frac{1}{\lvert V\rvert}\frac{\mathbf{1}_{\{u,v\}\in E}}{deg(u)}$	$= \rho(u,v)$

Voter and Invasion dynamics identical iff constant degree graph.

Reverse Chains

How did I get my opinion?

Reverse Chains

- Whose opinion at the previous time step u has now?
  - It is v's opinion if (v, u) was sampled.
  - It is u's opinion if $(\cdot, u)$ was not sampled.
- This gives a MC on V which is tracing the opinions back in time. It has a transition function q, given by

$$q(u, v) = \begin{cases} \rho(v, u) = \rho(v|u)\rho_2(u) & v \neq u \\ 1 - \rho_2(u) & v = u, \end{cases}$$

  where $\rho_2(u) = \sum_v \rho(v, u)$ is the second marginal of ρ.

Example 4 (Reverse Chains)

<u>Voter</u>

RW on V:

- ρ_2 is uniform on V; and
- $\rho(\cdot|u)$ uniform on neighbors of u.

<u>Invasion</u>

Conditioned on a transition, probability is reciprocal to degree of target vertex:

- $\rho_2(u) = \frac{1}{|V|} \sum_{\{u', u\} \in E} \frac{1}{deg(u')}$; and
- $\rho(\cdot|u) = \frac{\frac{1}{deg(\cdot)}}{\sum_{\{u', u\} \in E} \frac{1}{deg(u')}}$.

Duality with Reverse Opinion Flow

Initial opinion distribution and the flow of opinions back in time determine the distribution of the process. This flow is a family $\mathbf{Z}$ of coalescing chains:

Definition 3 (Reverse Flow/Coalescing Reverse Chains)

Let $\mathbf{Z} = (\mathbf{Z}_t(u) : u \in V, t \in \mathbb{Z}_+)$ *be the process*

- *For* $u \in V$*, set* $Z_0(u) = u$*.*
- *At* $t \in \mathbb{Z}_+$*, sample* $(\mathcal{V}, \mathcal{U})$ *according to* ρ*.*
- *AT time* $t+1$*, set all chains currently in* $\mathcal{U}$ *at time* t *to* $\mathcal{V}$ *and keep all others where they are.*

$$Z_{t+1}(u) = \begin{cases} \mathcal{V} & \text{if } Z_t(u) = \mathcal{U} \\ Z_t(u) & \text{otherwise} \end{cases}$$

Interpretation

- For $u \in V$, $(Z_t(u) : t \in \mathbb{Z}_+)$ is a MC with TF q starting from u, and which represents (in distribution) the vertex whose opinion t units back in time u currently holds.
- The same holds jointly over $u \in V$ and $t \in \mathbb{Z}_+$.
- When $Z_\cdot(u)$ and $Z_\cdot(u')$ meet, they **coalesce**. In terms of opinion flow: the opinion lineage for u and u' from that point backward in time is the same.

Note

This duality is well-known and documented for continuous-time Voter model: see Durrett (1988); Aldous and Fill (2002); Oliveira (2012) and references therein.

Coincidence of Tail Behavior

Why reverse flow?

- Past: key tool for analysis of Voter model (mostly on infinite state spaces like $\mathbb{Z}^d$) for getting probability of consensus, distribution of time for absorption, joint distribution of opinions at pairs or more vertices, etc.
- Our work: Access to λ by reducing the eigenvalue problem to tails of coalescence time of two reverse chains.

Let

$$\sigma_{u,u'} = \inf\{t \in \mathbb{Z}_+ : Z_t(u) = Z_t(u')\} \qquad \text{(coalesence time of } u, u'\text{)}$$
$$\sigma = \max_{u,u'} \sigma_{u,u'} \qquad \text{(coalescence time of } \mathbf{Z}\text{)}$$
$$\lambda_{CMC} = \lim_{t\to\infty}(P(\sigma > t))^{1/t} \qquad \text{(geometric tail of } \sigma\text{)}$$

Before we continue, we recall (Proposition 2) that the QSD ν is a left eigenvactor for $p|_\Omega$ corresponding to the Perron eigenvalue λ:

$$\nu p|_\Omega = \lambda\nu$$

Theorem 5

Under General Evolution, Definition 2:

1. $\lambda_{CMC} = \max_{u,u'} \lim_{t\to\infty} (P(\sigma_{u,u'} > t))^{1/t} = \lim_{t\to\infty} (\max_{u,u'} P(\sigma_{u,u'} > t))^{1/t}$.
2. $\lambda = \lambda_{CMC}$.

Example: Complete Graph

Example 6 (Voter on Complete Graph)

Consider the Voter model on K_n, the complete graph with n vertices.

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- For any two distinct vertices u, u'

$$\sigma_{u,u'} \sim \mathrm{Geom}(\frac{2}{n(n-1)}) \implies \lambda = \lambda_{CMC} = 1 - \frac{2}{n(n-1)}. \qquad (4)$$

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- To calculate the QSD, identify all states with $j = 1, \dots, n-1$ "yes" opinions as a single class, and write $\nu(``j")$ for probability of this class (of $\binom{n}{j}$ states).

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- Class "j" can be reached from either "$j-1$", "j", "$j+1$" with respective transition probabilities:

$$\frac{(j-1)(n-j+1)}{n(n-1)}, \frac{j(j-1)}{n(n-1)} + \frac{(n-j)(n-j-1)}{n(n-1)}, \frac{(j+1)(n-j-1)}{n(n-1)}.$$

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- Thus, the sum of the "j"-th column of $p|_\Omega$ is

$$\frac{(j-1)(n+1)}{n(n-1)} + \frac{(n-j-1)(n+1)}{n(n-1)} = \frac{(n-2)(n+1)}{n(n-1)}$$
$$= \frac{(n-1)n - n + (n-2)}{n(n-1)} = 1 - \frac{2}{n(n-1)} = \lambda.$$

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- Concluding: the column-sum is independent of "j", which implies that ν, the left eigenvector corresponding to λ is constant, or:

$$\nu(\text{"}j\text{"}) = \frac{1}{n-1}.$$

Example: Voter on the Cycle

Example 7 (Voter model on the Cycle)

Consider now the Voter model on the cycle $\mathbb{Z}_N$.

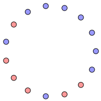

Figure: Initial opinion assignment

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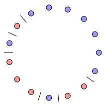

Figure: Interfaces between opinions

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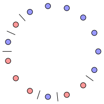
- The evolution corresponds to movement of these interfaces. Each step either:



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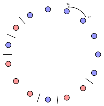

Figure: None of interface move

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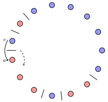

Figure: An interface moves

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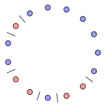

Figure: An interface move, completed

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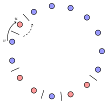

Figure: Interfaces cancel each other

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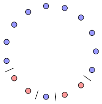

Figure: Interfaces cancel each other, completed

Example: Voter on the Cycle

Example 7 (Voter model on the Cycle)

Consider now the Voter model on the cycle $\mathbb{Z}_N$.

- Each non-absorbing state induces an even number of interfaces between "yes" and "no".
- The evolution corresponds to movement of these interfaces. Each step either:
  - None of the interfaces move.
  - One interface moves in either direction equally likely; and
  - When two interfaces meet, they are both eliminated.
- The number of interfaces will eventually reach two.

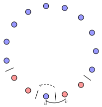

Figure: Down to two interfaces

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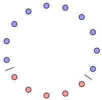

Figure: Down to two interfaces, completed

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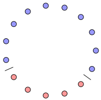

Figure: Down to two interfaces, completed

Conclusions

- The QSD is supported on the states with exactly two interfaces.

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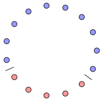

Figure: Down to two interfaces, completed

Conclusions

- The QSD is supported on the states with exactly two interfaces.
- Under $p|_\Omega$, the number of "yes" between the two interfaces performs a symmetric RW, absorbed at 0 and N.

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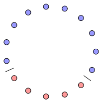


Figure: Down to two interfaces, completed

Conclusions

- The QSD is supported on the states with exactly two interfaces.
- Under $p|_\Omega$, the number of "yes" between the two interfaces performs a symmetric RW, absorbed at 0 and N.
- A comeback! The QSD is the same as for the RW from Example 2.

Example: Cycle, continued

Recall the QSD from Example 2:

$$\nu(x) = \tan(\frac{\pi}{2N})\sin(\frac{x\pi}{N}), \;\; x = 1, \ldots, N-1.$$

What we actually proved is

Proposition 2

The QSD for the Voter model on $\mathbb{Z}_N$ is a rotationally invariant distribution on "yes" and "no" opinions with a single contingent cluster of "yes" opinions distributed according to ν.

Questions

- What about QSD for system conditioned to have more than two contingent clusters?
- $\mathbb{Z}_N \times \mathbb{Z}_N$?

Complete Bipartite Graph

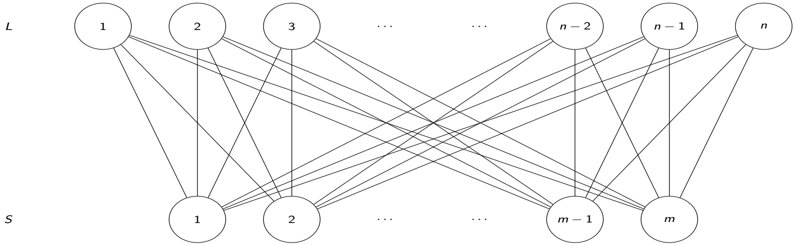
Setup

G is the **complete bipartite graph** $K_{m,n}$:

- V: the disjoint union of **partitions** S and L, $|S| = m$, $|L| = n$;
- E: all sets of the form $\{s, \ell\}$, $s \in S, \ell \in L$.

Note

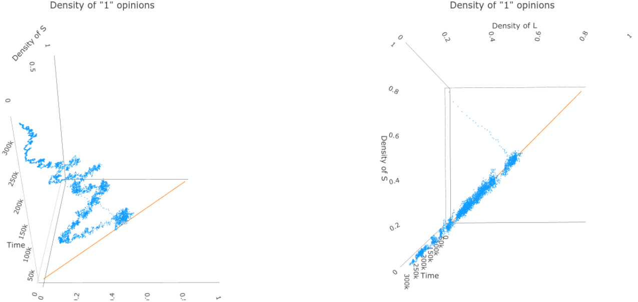
- Rudimentary network with a small number of highly connected agents, and a large number of agents with low connectivity.
- Extensive literature on Voter model and very little on QSD for model, so good place to start, I guess.
- In our work: m fixed while $n \to \infty$.



Voter on Complete Bipartite

Motivation and simulations

Our work was motivated by a series of lectures by Sidney Redner given in NetSci2019. The version of the following simulation presented at the talks is what started it for me.



Simulation of the Voter model on $K_{200,1000}$.

Observation

Starting from all "yes" in S, "no" in L, the system rapidly enters a long period ("metastable") where the proportions are roughly the same, where it performs a RW that is slowed down near consensus.

Question

QSD?

Voter on Complete Bipartite

MCREU ‘19 with Hugo Panzo and then undergrads Philip Speegle and Oliver Vandenberg

Proposition 3

Consider the Voter model on $K_{m,n}$. Then

$$\lambda_{CMC} = \lambda_{n,m} = 1 - \frac{2}{n+m}\left(1 - \sqrt{1 - \frac{1}{2n} - \frac{1}{2m}}\right) = 1 - \frac{\gamma_{m,n}}{n+m}. \tag{5}$$

Note

- Proof relies on the simple structure of the graph and the reverse chains, which reduces the calculation to a two-state Markov chain (states: walks are in same or different partition).
- $$\gamma_m = \lim_{n\to\infty} \gamma_{m,n} = 2\left(1 - \sqrt{1 - \frac{1}{2m}}\right) \underset{m\to\infty}{\sim} \frac{1}{2m}. \tag{6}$$
- The exact form in (5) is crucial for studying the QSD as $n \to \infty$ because the QSD eigenvector equation in Proposition 1 reduces to a triviality.

Sibuya: Discrete Heavy Tailed Distributions

Sibuya Distributions

The **Sibuya distribution** with parameter $\gamma \in (0,1)$. This is a probability distribution on $\mathbb{N}$ with generating function ϕ_γ and PMF $p_\gamma(z)$ given by

$$\begin{aligned}\phi_\gamma(z) &= 1-(1-z)^\gamma \\ &= \sum_{k=1}^{\infty} \underbrace{\frac{(k-1-\gamma)(k-2-\gamma)\cdots(1-\gamma)\gamma}{k!}}_{=p_\gamma(k)} z^k\end{aligned}$$

As a result, $Sib(\gamma)$ is heavy-tailed with

$$p_\gamma(k) \sim c_\gamma \frac{1}{k^{\gamma+1}}, \;\; c_\gamma = \frac{\sin(\gamma\pi)}{\pi}\Gamma(1+\gamma).$$

QSD Asymptotics for Voter

Theorem 8

Let $C \sim Bern(\frac{1}{2})$ and $D \sim Sib(\gamma_m)$ be independent with

$$\gamma_m \stackrel{(6)}{=} 2\left(1 - \sqrt{1 - \frac{1}{2m}}\right).$$

Then the QSD for the Voter model on $K_{m,n}$ converges as $n \to \infty$ to:

1. *All vertices of S take opinion C.*
2. *All but D vertices of L have opinion C.*

Note

- Exact formula for λ_{CMC} from Proposition 3 is key to analysis, and leads to showing that S reaches consensus. With this
- QSD equation essentially reduces to difference equation for dissenting opinions in L.

QSD Asymptotics for Invasion

with MCREU '20 undergrads Van Hovenga and Edith Lee

This is where a hybrid system shows up!

Two differences

- No nice closed-form expression for λ_{CMC}: the reverse chain reduces to a three state chain with nasty characteristic polynomial.
- And, *no consensus on either partitions*. While in Voter, S reaches consensus, here it keeps changing nearly all the time.

Proposition 4

Consider the Invasion process on $K_{m,n}$. Then

$$\lambda_{CMC} = 1 - \frac{2m}{n^2(n+m)} + o(n^{-3}) \tag{7}$$

Note

- The proof of the proposition is based on the Taylor expansion for the Perron eigenvalue, and the first nontrivial term is obtained from the Hessian.
- Absorption times under QSD are Geom$(1-\lambda)$. Expectations:
  - Invasion: $\sim \frac{n^3}{2m}$.
  - Voter, (5),(6): $\approx 2mn$ (when m is also large).

QSD Asymptotics for Invasion

with MCREU '21 undergrads Clay Allard, Shrikant Chand and Julia Shapiro

Switch from *counting opinions* in L to *proportions of opinions*, leading to the introduction of $\bar{\nu}$ on $\{0,\dots,m\}\times[0,1]$:

$$\bar{\nu}(k,dx)=\nu(k,nx)\delta_{\{0,\dots,n\}}(nx) \qquad (8)$$

We have the following:

Theorem 9

Consider the Invasion model on $K_{m,n}$. Then as $n\to\infty$,

$$\bar{\nu}(k,dx)\Rightarrow\binom{n}{k}x^k(1-x)^{m-k}dx. \qquad (9)$$

In particular,

1. *At the limit both marginals are uniform on $\{0,\dots,m\}$ and $[0,1]$, respectively.*
2. *The first marginal conditioned on the second equals x: $Bin(m,x)$.*
3. *The second marginal conditioned on the first equals k: $Beta(k+1,m-k+1)$.*

Observations

- In contrast to Voter, here QSD is very far from consensus.
- The second marginal corresponds to the QSD for the Wright-Fisher diffusion generator $\frac{1}{2}x(1-x)\frac{d^2}{dx^2}$ on $[0,1]$ absorbed on the boundary.
- A hybrid system emerges: first-order terms give the discrete part, and higher (smaller) order terms give the continuous part.

Derivation of QSD for Invasion

Step 1. Rearrangement

The QSD equation (1) can be rearranged as

$$S(k,l) + L(k,l) - \mathbf{1}_{\Delta}(k,l)(S(0,0)+L(0,0)) = (\lambda-1)\nu(k,l) \quad \text{with}$$

$$\sum_k S(k,l) = 0 \qquad \sum_l L(k,l) = 0.$$

- $S(k,l)$: Associated with $(v,u) \in L \times S$, probability $\frac{n}{n+m} \sim 1$
- $L(k,l)$: Associated with $(v,u) \in S \times L$, probability $\frac{m}{n+m} = O(n^{-1})$
- Capturing absorption
- Adjustment so L and S can be expressed as sums of differences. From Proposition 4, $\lambda - 1 \sim \frac{2m}{(m+n)n^2}$.

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Derivation of QSD for Invasion

Step 1. Rearrangement

The QSD equation (1) can be rearranged as

$$S(k,l) \quad +L(k,l) + \frac{1}{2}(\lambda-1)\mathbf{1}_{\Delta}(k,l) = (\lambda-1)\nu(k,l) \quad \text{with}$$

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- Capturing absorption
- Adjustment so L and S can be expressed as sums of differences. From Proposition 4, $\lambda - 1 \sim \frac{2m}{(m+n)n^2}$.
- Result of summing up both sides

Derivation of QSD for Invasion

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- Capturing absorption
- Adjustment so L and S can be expressed as sums of differences. From Proposition 4, $\lambda - 1 \sim \frac{2m}{(m+n)n^2}$.

Step 2. Switch to proportions on L

Replace second component by proportion, giving

$$\int f(k,x)d\bar{S}(k,x) + \int f(k,x)d\bar{L}(k,x) + \frac{\lambda-1}{2}(f(0,0)+f(m,1)) = (\lambda-1)\int f(k,x)d\bar{\nu}$$

$$\int \bar{S}(dk,x) = 0 \qquad \int \bar{L}(k,dx) = 0 \tag{10}$$

Derivation of QSD for Invasion

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Step 3. First component

As $n \to \infty$,

- Take subsequential limit for measures $\bar{\nu}$ (think of it as $m+1$ measures $\bar{\nu}(\cdot, dx)$ converging simultaneously).

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As $n \to \infty$,

- Take subsequential limit for measures $\bar{\nu}$ (think of it as $m+1$ measures $\bar{\nu}(\cdot, dx)$ converging simultaneously).
- Denote the limit by $\bar{\nu}_\infty(\cdot, dx)$ and write $\bar{\nu}_{\infty,2}$ for the marginal of the second component.

Derivation of QSD for Invasion

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- In (10):
  - All terms but integral WRT to $\bar{S}$ vanish.
  - Forcing integral WRT to $\bar{S}$ to be equal to zero.

Derivation of QSD for Invasion

Step 2. Switch to proportions on L

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Derivation of QSD for Invasion

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- In (10):
  - All terms but integral WRT to $\bar{S}$ vanish.
  - Forcing integral WRT to $\bar{S}$ to be equal to zero.
- Using explicit form of $\bar{S}$ in terms of $\bar{\nu}$, this leads to a recurrence relation for the sequence $k \to \bar{\nu}_\infty(k,dx)$.

Derivation of QSD for Invasion

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Replace second component by proportion, giving

$$\int f(k,x)d\bar{S}(k,x) \quad + \int f(k,x)d\bar{L}(k,x) \quad + \frac{\lambda-1}{2}(f(0,0)+f(m,1)) = (\lambda-1)\int f(k,x)d\bar{\nu}$$

$$\int \bar{S}(dk,x) = 0 \qquad \int \bar{L}(k,dx) = 0 \tag{10}$$

Step 3. First component

The unique solution is

$$\bar{\nu}_\infty(k,dx) = \binom{m}{k} x^k (1-x)^{m-k} \bar{\nu}_{\infty,2}(dx) \tag{11}$$

equivalently $\nu_\infty(dk|x) \sim \mathrm{Bin}(m,x)$ under any subsequential limit.

Derivation of QSD for Invasion, Continued

Step 3 gave $\bar{\nu}_\infty(dk|x) \sim \text{Bin}(m, x)$. It's left to determine $\bar{\nu}_{\infty,2}$.

Step 4. Second Component

More work: To access distribution of second component need to eliminate terms of higher order of magnitude.

- Use a smooth bounded test function $f = f(x)$ in the representation (10) to eliminate integral WRT to $\bar{S}$.

Derivation of QSD for Invasion, Continued

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Step 4. Second Component

More work: To access distribution of second component need to eliminate terms of higher order of magnitude.

- Use Taylor expansion for f and explicit form for $\bar{L}$ to obtain

$$\int f(x)d\bar{L} = \frac{1}{(m+n)n}\int (k(1-x)-(m-k)x)\, f'(x)d\bar{\nu} + \frac{1}{2(m+n)n^2}\int (k(1-x)+(m-k)x)\, f''(x)d\bar{\nu} + o(n^{-3})$$

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- Curse: remaining terms in (10) are $O(n^{-3})$ so only conclusion is that first integral on RHS is equal to some $c_n(f)$ which tends to 0.

Derivation of QSD for Invasion, Continued

Step 3 gave $\bar{\nu}_\infty(dk|x) \sim \text{Bin}(m, x)$. It's left to determine $\bar{\nu}_{\infty,2}$.

Step 4. Second Component

More work: To access distribution of second component need to eliminate terms of higher order of magnitude.

- Use Taylor expansion for f and explicit form for $\bar{L}$ to obtain

$$\int f(x) - \underbrace{c_n}_{=o(1)} x d\bar{L} = \frac{1}{2(m+n)n^2} \int (k(1-x) + (m-k)x) f''(x) d\bar{\nu} + o(n^{-3})$$

- Blessing: subtract a linear term of the form $c_n x$ from f to eliminate that first integral.

Derivation of QSD for Invasion, Continued

Step 3 gave $\bar{\nu}_\infty(dk|x) \sim \mathrm{Bin}(m,x)$. It's left to determine $\bar{\nu}_{\infty,2}$.

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- Plugging this into (10) and multiplying by $2(m+n)n^2$ we obtain

$$\int (k(1-x) + (m-k)x) f''(x) d\bar{\nu} + o(n^{-1})$$
$$= 2(m+n)n^2(\lambda - 1)\left(\int f - c_n x d\bar{\nu} + \frac{f(0) + f(1) - c_n}{2}\right).$$

Derivation of QSD for Invasion, Continued

Step 3 gave $\bar{\nu}_\infty(dk|x) \sim \text{Bin}(m, x)$. It's left to determine $\bar{\nu}_{\infty,2}$.

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More work: To access distribution of second component need to eliminate terms of higher order of magnitude.

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- From Proposition 4, $(\lambda - 1) \sim -2m(m+n)^{-1}n^{-2}$

Derivation of QSD for Invasion, Continued

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Step 4. Second Component

More work: To access distribution of second component need to eliminate terms of higher order of magnitude.

- Plugging this into (10) and multiplying by $2(m+n)n^2$ we obtain

$$\int (k(1-x) + (m-k)x) f''(x) d\bar{\nu} + o(n^{-1})$$
$$= -4m\left(\int f - c_n x d\bar{\nu} + \frac{f(0) + f(1) - c_n}{2}\right) + o(1).$$

  - From Proposition 4, $(\lambda - 1) \sim -2m(m+n)^{-1}n^{-2}$
  - $c_n = o(1)$, so clean a little

Derivation of QSD for Invasion, Continued

Step 3 gave $\bar{\nu}_\infty(dk|x) \sim \text{Bin}(m,x)$. It's left to determine $\bar{\nu}_{\infty,2}$.

Step 4. Second Component

More work: To access distribution of second component need to eliminate terms of higher order of magnitude.

- Plugging this into (10) and multiplying by $2(m+n)n^2$ we obtain

$$\int (k(1-x)+(m-k)x)\, f''(x)d\bar{\nu}$$
$$= -4m\left(\int fd\bar{\nu} + \frac{f(0)+f(1)}{2}\right) + o(1).$$

  - From Proposition 4, $(\lambda - 1) \sim -2m(m+n)^{-1}n^{-2}$
  - $c_n = o(1)$, so clean a little
  - From (11), $\int k\bar{\nu}_\infty(dk|x) = mx$.

Derivation of QSD for Invasion, Continued

Step 3 gave $\bar{\nu}_\infty(dk|x) \sim \mathrm{Bin}(m, x)$. It's left to determine $\bar{\nu}_{\infty,2}$.

Step 4. Second Component

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- Plugging this into (10) and multiplying by $2(m+n)n^2$ we obtain

$$2\int x(1-x)f''(x)\bar{\nu}_{\infty,2}(dx)$$
$$= -4m\left(\int fd\bar{\nu} + \frac{f(0)+f(1)}{2}\right).$$

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- Rearranging, end up with

$$\int x(1-x)f''(x) + 2f(x)d\bar{\nu}_{\infty,2}(x) = 2(f(0)+f(1)), \tag{12}$$

for all subsequential limits.

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- In conclusion, we showed that for subsequential limit,

$$\bar{\nu}_\infty(k, dx) = \binom{m}{k} x^k (1-x)^{m-k} dx.$$

  Thus a limit exists and is of the form above.

Derivation of QSD for Invasion, Continued

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$\square$

Thank you. Special thanks to organizers.