(Efficient Coupling for) Diffusion with Redistribution

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[1] based on joint work with H. Panzo and E. Tripp

Introduction

Model

- Diffusion on a bounded domain $D \subset \mathbb{R}^d$
- When hits the boundary, the process is redistributed back into the domain, starting the diffusion afresh.

  Specifically, if $z \in \partial D$ is hit, then the process stars from a location sampled from a probability distribution μ_z, with $\mu_z(D) = 1$.
- Mechanism is repeated indefinitely.

Why ?

- Fleming-Viot interacting particle model – also as a method to sample quasi-stationary distribution for BM.
  [BHM00] (more recently [BBF12] [BBP12] [GK12] [GK14])
- Then "mean-field" version, that grew into this model and raised many questions.
  [GK02] [GK04][GK07] [BP07] [BP09] [LLR08] [KW11] [KW11a][PL12] [B14]
- Corresponds to analysis of a differential operator with non-local boundary conditions – semiclassical analysis.
- Potential application: Finance ?

Main question

Long-run behavior, convergence to stationarity.

(some difficulties: not Feller, never reversible, for FV existence for infinite time horizon was open until recently)

Long-run behavior

Let's denote our process by $X = (X_t : t \geq 0)$.

Here t represents time and X_t is the location at time t.

Run the process for a long time... what happens ?

From general principles (Doeblin condition) one can show that

- The process is ergodic: the distribution of X_t converges to a limit independent of the distribution at time 0, the stationary distribution.
- The process is exponentially ergodic: the convergence occurs at an exponential rate.

Characterization of the exponential rate is more subtle.

Stationary distribution

Assumptions.

- The domain D is smooth.
- The underlying diffusion is generated by the elliptic operator L ($L = \frac{1}{2}\Delta$ for Brownian motion)
- The mapping $D \ni z \to \mu_z$ is continuous (in the weak topology)

Characterization of Stationary distribution

Proposition 1 (B-Pinsky, [BP09])

1. *Sequence of points X hits ∂D is a pos. recur. MC on ∂D with ! stat. dist. m.*
2. *X has ! stationary distribution π and*

$$d\pi(y) = C^{-1} \int_{\partial D} \int_D G(x, y) d\mu_z(x) dm(z) dy,$$

where G is the Green's function for L, and C a normalizing constant.

Exponential Ergodicity

The total variation distance between two probability measures on the same measure space is defined as

$$\|Q - Q'\|_{TV} = \sup_A \left(Q(A) - Q'(A)\right).$$

Theorem 1 (B-Pinsky, [BP09])

Let $\mathcal{L}$ be the restriction of L to

$$\{u : \forall z \in \partial D, \ \lim_{y \to z} u(z) = \int u(x) d\mu_z(x)\},$$

and

$$\gamma_1(\mu) = \min\{Re(\lambda) : 0 \neq \lambda \textit{ eigenvalue for } -\mathcal{L}\}.$$

Then

$$-\frac{1}{t} \sup_x \ln \|P_x(X_t \in \cdot) - \pi\|_{TV} \underset{t \to \infty}{\to} \gamma_1(\mu) \in (0, \infty).$$

Remarks

- The operator $\mathcal{L}$ can be viewed as the generator of the diffusion with redistribution, and γ is referred to as the spectral gap.
- The theorem was previously proved by Grigorescu and Kang for a specific case (BM redistributed to a fixed point).
- The proof is analytical and relies on analysis of semigroups (difficulties: semigroup not strongly continuous, and $\mathcal{L}$ not densely defined)

1D

Assumptions

- Domain is the unit interval $D = (0, 1) \subset \mathbb{R}$
- The underlying diffusion is Brownian Motion with generator

$$L = \frac{1}{2}\frac{d^2}{dx^2}.$$

Dirichlet eigenvalues and exit times

- Let $\lambda_0^D(\ell)$ denote the first Dirichlet eigenvalue for $-L$ on $(-\frac{\ell}{2}, \frac{\ell}{2})$:

$$\lambda_0^D(\ell) = \frac{\pi^2}{2\ell^2}.$$

- Let $T(\ell)$ exit time of BM from $(-\frac{\ell}{2}, \frac{\ell}{2})$, starting at 0.
- Then $T(\ell)$ has exponential tail $\lambda_0^D(\ell)$:

$$P(T(\ell) > t) \sim ce^{-\lambda_0^D(\ell)t}.$$

Results on γ_1

Deterministic redistribution

Solving the eigenvalue problem in Theorem 1 gives the convergence rates:

Corollary 1

$$\gamma_1(\delta_a, \delta_b) = \lambda_0^D(L_0(a,b)),$$

where

$$L_0(a,b) = \frac{1}{2}\max\{a, 1-b, 1+b-a\}.$$

In particular,

$$\lambda_0^D(1) = \lim_{a \to 0^+} \gamma_1(\delta_a, \delta_{1-a}) < \gamma_1(\delta_a, \delta_b) \leq \gamma_1(\delta_{\frac{2}{3}}, \delta_{\frac{1}{3}}) = \lambda_0^D(\frac{1}{3}).$$

Equal redistribution measures

By Fourier techniques it was shown that

Theorem 2 (Li-Leung-Rakesh [LLR08])

If $\mu_0 = \mu_1$, *then*

$$\gamma_1 = \lambda_0^D(1/2).$$

Both results do not provide any intuition as for why they hold.

Probabilistic approach to exponential ergodictiy: coupling

Definition 1

- *A coupling for the generator $\mathcal{L}$ is a process (X, Y) such that marginal processes X and Y are each generated by $\mathcal{L}$.*
- *The coupling is Markovian if each of the marginals is Markov processes with respect to the filtration generated by (X, Y).*

The coupling time τ is defined as

$$\tau = \inf\{t \geq 0 : X_t = Y_t\}$$

Lemma 3 (Coupling inequality)

$$d_t(x, y) := \|P_x(X_t \in \cdot) - P_y(X_t \in \cdot)\|_{TV} \leq P_{x,y}(\tau > t).$$

Let $d_t = \sup_{x,y} d_t(x, y)$. Then from Theorem 1, we have that

$$\frac{1}{t} \ln d_t \to -\gamma_1(\mu).$$

Definition 2

A coupling is efficient if

$$\lim \frac{1}{t} \ln P_{x,y}(\tau > t) \leq \lim \frac{1}{t} \ln d_t(x, y).$$

Efficiency converts spectral problem into an absorption problem.

Why coupling ?

1. Intuitive pathwise explanation to analytic results, at least in 1D case.
2. Sharper bounds than those previously obtained by analytical methods.
3. Also, it well known that for 1D diffusions, all successful order preserving couplings are efficient.

   In our model order cannot be always preserved.

Coupling for BM with redistribution I

Equal Redistribution measures
Recall that when $\mu_0 = \mu_1$, $\gamma_1 = \lambda_0^D(\frac{1}{2})$.

We are looking for an explanation. It is provided through the following.

Theorem 4 (Kolb-Wubker [KW11])

Suppose that $\mu_0 = \mu_1$. Let $\rho > 0$ denote the distance of the support of μ_0 from $\{0, 1\}$. If $x, y \in (0, 1)$ with $|y - x| < \rho$, then there exists an efficient coupling with $(X_0, Y_0) = (x, y)$ such that τ is dominated by the sum of 5 independent copies of $T(1/2)$.

One interesting feature of the coupling is that it is not Markovian. We do not know whether an efficient Markovian coupling even exists.

Proof of Theorem 4

Here is a sketch of the coupling.

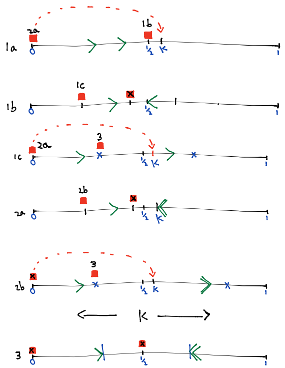


Figure: Stages of coupling

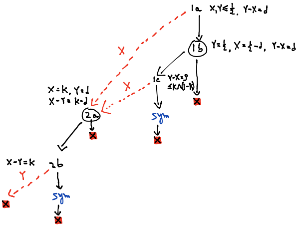


Figure: Flowchart for stages

Remark
What breaks the Markovian is the redistribution of Y from 0 in 2b, which we choose to be identical to that of X in 2a.

Coupling for BM with redistribution II

What about $\mu_0 \neq \mu_1$?

We already know the rates when $\mu_0 = \delta_a$ and $\mu_1 = \delta_b$. They are given by:

$$\gamma_1(\delta_a, \delta_b) = \lambda_0^D(L_0) \text{ where } L_0 = L_0(a, b) = \max \frac{1}{2}\{a, 1-b, 1+b-a\}.$$

Again, what is the explanation ?

Theorem 5 (B. - Panzo - Tripp [BPT])

For $x, y \in (0,1)$ with $0 < y - x \leq \min\{a, 1-b\}$ there exists a Markovian efficient coupling with $(X_0, Y_0) = (x, y)$ such that τ is dominated by the sum of $\lfloor 6 + 1/\min\{a, 1-b\} \rfloor$ independent copies of $T(L_0)$.

Remarks

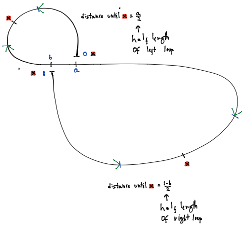
- We did this for random walk, which is a little messier with details.
- Coupling construction identifies L_0 as a geometric "bottleneck".
- The main difference and difficulty is that we cannot guarantee coupling when both copies are redistributed at the same time.
- The number of copies depends on a and b, in contrast with a uniform bound (5) in Theorem 4.
- This coupling is Markovian.
- Challenge: μ_0, μ_1 not deterministic.

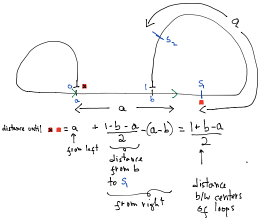
The bottleneck

We consider the state space as two loops:

- Left loop, $(0, a]$; and
- Right loop $[b, 1)$

There are two kinds of bottlenecks.





The copies start in the same loop and are symmetric with respect to its center. Use mirror coupling.

Effective length: $\frac{a}{2}$ or $\frac{(1-b)}{2}$.

The distance is the length of the short loop. Use translation coupling.

Effective length: $\frac{1+b-a}{2}$.

Remark

- The challenge was to show that these are the worst-case scenarios the coupling can achieve.

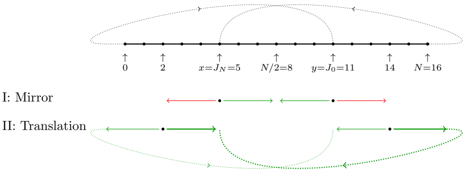
Observation: polynomial correction to rate

Setup

- $a = \frac{2}{3}$, $b = \frac{1}{3}$, $x = \frac{1}{3}$ and $y = \frac{2}{3}$.

Stages of coupling

- Mirror coupling until meeting or distance is $\frac{2}{3}$.
- Translation coupling until either hits boundary.



- It can be shown that $d_{x,y}(t) = P_{x,y}(\tau > t)$.
- But

$$
\begin{aligned}
F_\tau &= \text{P meeting at end of mirror} \times \text{DF of time for mirror} \\
&\quad + \text{P not meeting at end of mirror} \times (\text{ DF of time for mirror} * \text{DF of time for translation}) \\
&= \frac{1}{2} F_{T(1/3)} + \frac{1}{2} F^{*2}_{T(1/3)}
\end{aligned}
$$

$F^{*2}_{T(1/3)}$ has exponential tail with linear correction.

- Therefore $d_t(x, y) \sim cte^{-\lambda_0^D(1/3)t}$.

Bounds on γ_1

We proved the following in the generality of Theorem 1

Theorem 6 (B-Pinsky [BP07])

If $-\mathcal{L}$ possess a real non-zero eigenvalue, with minimal real part among all non-zero eigenvalues, then $\gamma_1(\mu) > \lambda_0^D$, the first Dirichlet eigenvalue of $-L$ on D.

Using this, it was shown that for the 1D BM

Theorem 7 (Li-Leung-Rakesh [LLR08])

$$\gamma_1(\mu_0, \mu_1) > \lambda_0^D(1).$$

Remarks

- In the above paper it was also shown that the realness condition does not hold for all diffusions.
- In an unpulished manuscript Li and Leung showed that $\gamma_1(\mu_0, \mu_1) \leq \lambda_0^D(\frac{1}{3})$.

We asked whether the lower bound of Theorem 6 is universal.

Another coupling result

We asked whether $\gamma_1(\mu) > \lambda_0^D$.

Consider BM with drift h, that is

$$L = \frac{1}{2}\frac{d^2}{dx^2} + h\frac{d}{dx},$$

on $(0,1)$ and redistribution

$$\mu_0 = \mu_1 = \delta_{\frac{1}{2}}.$$

Recall that the first Dirichlet eigenvalue for $-L$ on $(-\ell/2, \ell/2)$ is

$$\lambda_0^D(h,\ell) = \lambda_0^D(\ell) + \frac{h^2}{2}.$$

We have the following:

Theorem 8 (Kolb-Wubker [KW11a][B14])

$$\gamma_1 = \lambda_0^D(1/4) \wedge \lambda_0^D(h, 1/2).$$

and there exists an efficient coupling.

Remarks

- Kolb and Wubker showed that $\gamma_1 = \lambda_0^D(\frac{1}{4})$ for large enough h, and conjectured the critical value.
- B. completed the picture.
- Both results are by coupling.
- This provides a counterexample, as $\lambda_0^D = \lambda_0^D(h, \frac{1}{2}) > \gamma_1$ for h large.

Thank you.

Bibliography for redistribution from boundary I

Iddo Ben-Ari

Coupling for Drifted Brownian Motion on an Interval with Redistribution from the Boundary.

Electronic Communications in Probability, 19(16), 1–11, 2014.

Iddo Ben-Ari, Hugo Panzo and Elizabeth Tripp

Efficient Coupling for Random Walk with Redistribution.

submitted.

Iddo Ben-Ari and Ross G. Pinsky.

Spectral analysis of a family of second-order elliptic operators with nonlocal boundary condition indexed by a probability measure.

J. Funct. Anal., 251(1):122–140, 2007.

Iddo Ben-Ari and Ross G. Pinsky.

Ergodic behavior of diffusions with random jumps from the boundary.

Stochastic Process. Appl., 119(3):864–881, 2009.

Mariusz Bieniek, Krzysztof Burdzy, and Sam Finch.

Non-extinction of a Fleming-Viot particle model.

Probab. Theory Related Fields, 153(1-2):293–332, 2012.

Bibliography for redistribution from boundary II

Mariusz Bieniek, Krzysztof Burdzy, and Soumik Pal.
Extinction of Fleming-Viot-type particle systems with strong drift.
Electron. J. Probab., 17:no. 11, 15, 2012.

Krzysztof Burdzy, Robert Hołyst, and Peter March.
A Fleming-Viot particle representation of the Dirichlet Laplacian.
Comm. Math. Phys., 214(3):679–703, 2000.

Ilie Grigorescu and Min Kang.
Brownian motion on the figure eight.
J. Theoret. Probab., 15(3):817–844, 2002.

Ilie Grigorescu and Min Kang.
Path collapse for multidimensional Brownian motion with rebirth.
Statist. Probab. Lett., 70(3):199–209, 2004.

Ilie Grigorescu and Min Kang.
Ergodic properties of multidimensional Brownian motion with rebirth.
Electron. J. Probab., 12: no. 48, 1299–1322, 2007.

Ilie Grigorescu and Min Kang.
Immortal particle for a catalytic branching process .
Probab. Theory Related Fields, 153(1-2), 333–361, 2012.

Bibliography for redistribution from boundary III

Ilie Grigorescu and Min Kang.
Markov Processes with Redistribution
to appear in *Markov Processes and Related Fields*

Martin Kolb and Achim Wübker.
On the spectral gap of Brownian motion with jump boundary.
Electron. J. Probab., 16:no. 43, 1214–1237, 2011.

Martin Kolb and Achim Wübker.
Spectral analysis of diffusions with jump boundary.
J. Funct. Anal., 261(7):1992–2012, 2011.

Yuk J. Leung and Wenbo Li.
Fastest rate of convergence for brownian motion with jump boundary.
In preparation.

Yuk J. Leung, Wenbo V. Li, and Rakesh.
Spectral analysis of Brownian motion with jump boundary.
Proc. Amer. Math. Soc., 136(12):4427–4436, 2008.

Jun Peng and Wenbo V. Li, *Diffusions with holding and jumping boundary*, Science China Mathematics (2012), 1–16 (English).