Poisson Boundary and Transformations of Markov Chains

Iddo Ben-Ari, University of Connecticut
work with Behrang Forghani

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Introduction

Setup

- $X = (X_n : n \in \mathbb{Z}_+)$ transient irreducible* MC on a countable state space S, and TF p.
- Transience $\Rightarrow \lim_{n\to\infty} X_n = \infty$, P_x-a.s. for all x.

Can be more specific than just "∞" ?

Sometimes, yes, in a natural way.

Example

Take $S = \mathbb{Z}$. For $\epsilon \in (0,1)$ consider

$$p(x,y) = \begin{cases} \frac{1}{2}(1+\epsilon(y-x)\mathrm{sgn}(x)) & |y-x| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then under P_x,

$$\lim_{n\to\infty} X_n = +\infty \text{ with probability } p_x \in (0,1),$$

and

$$\lim_{n\to\infty} X_n = -\infty \text{ with probabilility } 1 - p_x \in (0,1).$$

That is,

there exists a $\{+\infty, -\infty\}$-valued nondegenerate random variable X_∞, such that

$$\lim_{n\to\infty} X_n = X_\infty, \ P_x\text{−}a.s..$$

Let's try to develop some theory.

First observations

What do we want ?

A metric compactification of S, $\bar{S} = S \cup \partial\bar{S}$, such that there exists a $\partial\bar{S}$-valued random variable X_∞ so that

$$lim_{n\to\infty} X_n = X_\infty, \; P_x\text{-a.s., for all } x.$$

So...

- When one completes a metric space it is natural to consider the completion as equivalence classes of Cauchy sequences.
- Given a convergent sequence in $\bar{S}$ whose elements are in S, the definition of Cauchy sequence requires all sequences which are eventually the same to converge to same limit.
- The collection of events which do not distinguish between paths which are eventually the same is exactly the invariant σ-algebra $\mathcal{I}$.

$$\mathcal{I} = \{A \in \mathcal{F}_\infty : \theta^{-1}A = A\}.$$

- Thus, X_∞ attains values in the set of equivalence classes of paths which are eventually the same, and
- therefore for $A \in \mathcal{I}$, the event $\{X_\infty \in A\}$ is synonymous with the event A.

Now recall the following.

- A function $h : S \to \mathbb{R}$ is called p-harmonic if $h = ph$, that is

$$h(x) = \sum_y p(x,y)h(y).$$

  Equivalently, $h(x) = E_x[h(X_1)]$.
- If h is bounded, then h is p-harmonic if and only if $(h(X_n) : n \in \mathbb{Z}_+)$ is a martingale.

Poisson Boundary, take I

Let's take a look at the distribution of X_∞.

Fix $o \in S$ such that $o \to x$ for all $x \in S$.

- For $x \in S$, let μ_x denote the distribution of X_∞ under P_x. Then

$$\begin{aligned}\mu_x(A) &= P_x(X_\infty \in A) \\ &= P_x(A) = P_x(\theta^{-1}A) \\ &= \sum_y p(x,y)P_y(A).\end{aligned}$$

- That is, the function $x \to \mu_x(A) = P_x(A)$ is p-harmonic.
- Since $o \to x$, we have $n \in \mathbb{N}$ such that $p^n(o,x) > 0$. Thus, for every $A \in \mathcal{I}$,

$$P_o(A) \geq \sum_y p^n(o,y)P_y(A) \geq p^n(o,x)P_x(A)$$

$$\Rightarrow \mu_x \ll \mu_o$$

- Thus, $\mu_x(A) = \int_{\partial\bar{S}} \mathbf{1}_A(\zeta)K(x;\zeta)d\mu_o(\zeta)$, where $K(x;\cdot) = \frac{d\mu_x}{d\mu_o}$.
- Not hard to see (Funbini-Tonelli with the right choice of A), that $K(\cdot;\zeta)$ is p-harmonic (possibly except on a set of μ_o measure zero, but this can be eliminated).

Summarizing, $\partial\bar{S}$ is equipped with a probability measure μ_o and a collection of hamronic functions indexed by $\partial\bar{S}$, which together determine the probabilities of all invariant events.

The probability space $(\partial\bar{S}, \mu_o)$ is a first draft of the Poisson boundary.

Bounded harmonic functions

Recall that $d\mu_x = K(x;\cdot)d\mu_o$.

Let $H^\infty(p)$ denote the linear space of bounded p-harmonic functions.

Note that $H^\infty(p)$ is Banach space with the sup-norm.

Let $u \in H^\infty(p)$. Then

- $(u(X_n) : n \in \mathbb{Z}_+)$ is a bounded martingale.
- Therefore $\lim_{n\to\infty} u(X_n)$ exists and is $\mathcal{I}$-measurable.
- We can express the limit above as $f(X_\infty)$ for some $f \in L^\infty(\mu_o)$.

That is,

$$\begin{aligned} u(x) &= \lim_{n\to\infty} E_x[u(X_n)] = E_x[\lim_{n\to\infty} u(X_n)] \\ &= E_x[f(X_\infty)] = int_{\partial\bar{S}} f(\zeta)d\mu_x(\zeta) \\ &= \int_{\partial\bar{S}} f(\zeta)K(x;\zeta)d\mu_o(\zeta). \end{aligned}$$

Conversely, if $f \in L^\infty(\mu_o)$, then the RHS defines an element in $H^\infty(p)$.

Poisson Boundary, final version

We showed that for $u \in H^\infty(p)$, we have a representation

$$u(x) = \int_{\partial \bar{S}} f(\zeta) K(x;\zeta) d\mu_o(\zeta).$$

Is the presentation unique ?

A priori, no. Why ?

- We can have $\zeta \neq \zeta'$ such that $K(\cdot;\zeta') = K(\cdot;\zeta)$.

  Solution: quotient out.

- We can have ζ' such that $K(\cdot;\zeta)$ is it self a convex combination of other $K(\cdot;\zeta)$ or even an integral of them.

  Solution: restrict to those ζ such that $K(\cdot;\zeta)$ is minimal.

  A nonegative harmonic function u is minimal positive if whenever $0 \leq v \leq u$ is positive harmonic then $v = cu$ for some u.

  The point is that the the collection $K(\cdot;\zeta)$ is the set of extreme points of the compact linear space of positive harmonic functions on S equal to 1 at o, and therefore by Choquet's theorem, each element in this compact space is an integral (convex combination) of these elements with respect to a unique probability measure.

The resulting probability space is called the Poisson boundary. We will not change notation.

Example

1. Example from Slide 2.

Let's work backwards. Suppose that $u \in H^\infty(p)$.

- Let $\tau_\ell = \inf\{n \in \mathbb{Z}_+ : X_n = \ell\}$.
- Fix x and let $\ell_1, \ell_2 > 0$ such that $-\ell_1 < x < \ell_2$.
- Since $(u(X_n) : n \in \mathbb{Z}_+)$ is a bounded martingale, optional stopping gives

$$u(x) = E_x[u(X_{\tau_{-\ell_1} \wedge \tau_{\ell_2}})].$$
$$= u(-\ell_1) P_x(\tau_{-\ell_1} < \tau_{\ell_2}) + u(\ell_2) P_x(\tau_{-\ell_1} > \tau_{\ell_2}).$$

- Take $\ell_1 \to \infty$ then $\ell_2 \to \infty$ to obtain

$$u(x) = u(-\infty) P_x(\lim_{n\to\infty} X_n = -\infty) + u(+\infty) P_x(\lim_{n\to\infty} X_n = \infty).$$

  Note that the existence of the limits of u at $\pm\infty$ is part of the statement. Setting $o = 0$, the probabilities are both $\frac{1}{2}$.

- Summarizing:
  - $\partial \bar{S} = \{+\infty, -\infty\}$.
  - $\mu_o = \frac{1}{2}(\delta_{-\infty} + \delta_{+\infty})$.
  - $K(x; \zeta) = 2P_x(\lim_{n\to\infty} X_n = \zeta)$.
  - $Tu = (u(+\infty), u(-\infty))$.

Another example

2. RW on Abelian Group.

In general, if S is an Abelian group and p is a random walk on the group, then the Poisson boundary consists of a single point.

Here is a concrete example that is easy to prove (the technique usually does not work).

- Suppose that $S = \mathbb{Z}^d$ for $d \geq 3$.
- Let
$$p(x, y) = \begin{cases} \frac{1}{2d+1} & y = x \pm e_i \\ \frac{1}{2d+1} & y = x. \end{cases}$$
- Given $x = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d$, it is not hard to construct two copies of the RW on $\mathbb{Z}^d$, X^1, X^2 such that $X_0^1 = x$ and $X_0^2 = 0$, and which meet a.s. at some time $\tau < \infty$.
- Since $(u(X_n^1) : n \in \mathbb{Z}_+)$ ands $(u(X_n^2) : n \in \mathbb{Z}_+)$ are both bounded martingales, optional stopping gives
$$u(x) = E[u(X_\tau^1)] = E[u(X_\tau^2)] = u(0).$$

Conclusions of uniqueness

Once we have uniqueness, we have a mapping $T : H^\infty(p) \to L^\infty(\mu_o)$:

$$u(x) = \int_{\partial \bar{S}} K(x;\zeta)(Tu)(\zeta)d\mu_o(\zeta).$$

The mapping T is

1. Linear.
2. One-to-one.
3. Onto.
4. An isometry from $H^\infty(p)$, equipped with the sup-norm to $L^\infty(\mu_o)$.

Therefore $H^\infty(p)$ can be identified with $L^\infty(\mu_o)$.

This brings us to the main topic.

Motivation

In study of RW on groups one problem is to construct processes with given Poisson boundary, up to isomorphism of probability spaces.

In the context of general Markov chains isomorphism of probability spaces is too strong (e.g. Example 1. with different choices of o) This leads to the next question.

Problem. Find a mechanism for constructing MC with isomorphic space of bounded Harmonic functions.

To some extent, this is more natural from a probabilistic perspective: this is what we need to know in order to describe the limits at $+\infty$.

Transformed Markov Chains

Motivated by the work of Forghani and Kaimanovich on random walk on groups, the idea would be to construct new processes from X through stopping times.

We do this as follows:

- Let τ be a stopping time for X. That is,

$$\{\tau \le n\} \in \sigma(X_0, \ldots, X_n), \; n \in \mathbb{Z}_+.$$

- By iterating τ we can obtain a sequence of stopping times:

$$\tau_0 = \tau, \; \tau_{n+1} = \tau \circ \theta_{\tau_n}.$$

  Write $\langle \tau \rangle = \{\tau_0, \tau_1, \ldots\}$.

- With this sequence, one obtains a new process $Y = (Y_n : n \in \mathbb{Z}_+)$, defined through

$$Y_n = X_{\tau_n}.$$

- By the strong Markov property, this is also a Markov chain on S. with transition function q.

Immediate observation:

$$H^\infty(p) \subset H^\infty(q).$$

exam This follows from optional stopping theorem applied to the p-martingale $(u(X_n) : n \in \mathbb{Z}_+)$, when $u \in H^\infty(p)$.

What about the other inclusion ?

About the other inclusion.

It is usually false. Why ?

Example.

- $S = \mathbb{Z}$, and item
$$p(x,y) = \begin{cases} \frac{1}{2}(1+\epsilon) & y = x+1 \\ \frac{1}{2}(1-\epsilon) & y = x-1 \end{cases}$$
- This is a random walk on an abelian group so Poisson boundary is a single point, and $H^\infty(p)$ contains only constants (this can be proved directly with an argument similar to Example 1.)
- Take $\tau = 2$. Note that X_{2n+2} is either X_{2n} or $X_{2n} \pm 2$. Therefore
$$u(x) = \mathbf{1}_{2\mathbb{Z}}(x) \text{ and } \mathbf{1}_{2\mathbb{Z}+1}(x) \in H^\infty(q),$$
but not in $H^\infty(p)$.

So what is the "right" inclusion ?

Adding time to the mix

Consider now another transition function on $\mathbb{Z}_+ \times S$, obtained from p:

$$p+((m,x),(n,y)) = \delta_{n,m+1}p(x,y).$$

Let Z denote the corresponding MC. Here is a simple fact.

- The tail σ-algebra of X, $\mathcal{T}$ is equal to the invariant σ-algebra for Z, $\mathcal{I}^Z$.

To give a little feeling think of the event

$$\{X_{2n} \in C \text{ for all } n \text{ large enough}\}$$

- This is clearly in $\mathcal{T}$ but not in $\mathcal{I}$, as shifting the paths gives a new statement: not on odd times but on even times.
- For Z this is an invariant event because the "time" is included in the path, and shifting has no affect on the meaning.

Our result

Recall that $H^\infty(p) \subset H^\infty(q)$. We're trying to get some associated "reverse" inclusion. In general this is not possible.

The following theorem gives us a sufficient condition.

Theorem 1

Let ρ be a $\mathbb{Z}_+ v$ random variable. For each $n \in \mathbb{N}$, let

$$\rho_n = n + \rho \circ \theta_n.$$

If for all $x \in S$, $\limsup_{n\to\infty} P_x(\rho_n \in \langle \tau \rangle) = 1$, then

$$\mathcal{I}^Y \subset \mathcal{T}^X.$$

Idea of proof.

- if we can guess what large times are in $\langle \tau \rangle$ with asymptotic certainty just from the tail of the path, then the invariant σ-algebra of Y, which is determined by X evaluated at along $\langle \tau \rangle$ is in the tail σ-algebra of X.

Remarks. In our paper (soon on ArXiv) we provide an array of examples, including

- Deterministic τ.
- τ is hitting time of a recurrent set.
- Application for stopping times for semigroups, and randomized stopping time
- Example where condition does hold by $P_x(\rho_n \in \langle \tau \rangle) < 1$ for all x and all n.
- Example where condition cannot hold for any choice of ρ.

What Theorem says about harmonic functions

Corollary 1

Under the assumption in the theorem, $H^\infty(q)$ is isometrically embedded in $H^\infty(p+)$.

Proof.

- Consider $h \in H^\infty(q)$.
- Then $h(Y_n)$ converges P_x-a.s. to a $\mathcal{I}^Y$-measurable random variable $h(Y_\infty)$.
- From the theorem this is also in $\mathcal{T}^X = \mathcal{I}^Z$.
- Let $Th(n,x) = E^Z_{n,x}[h(Y_\infty)]$. Then
  - $Th \in H^\infty(p+)$.
  - Since $|h(Y_\infty)| \leq \|h\|_\infty$, $\|Th\|_\infty \leq \|h\|_\infty$.
  - But $Th(0,x) = h(x)$, therefore $\|Th\|_\infty \geq \|h\|_\infty$.

$\square$

Tighter inclusions

When is $H^\infty(p+)$ isometric to $H^\infty(p)$, namely when are all elements in $H^\infty(p+)$ constant in time ?

- If
$$\lim_{n\to\infty} \|p^n(x,\cdot) - p^{n+1}(x,\cdot)\|_{\mathrm{TV}} = 0, \text{ for all } x.$$
- The above condition is automatically satisfied if $p(x,x) = \frac{1}{2}$ (or any other positive constant) for all x.

When one of the conditions holds, then Corollary 1 gives
$$H^\infty(p) = H^\infty(q).$$

Obrigado. Grazie. Thank you.