School Policy Evaluated with Time-Reversible Markov Chains

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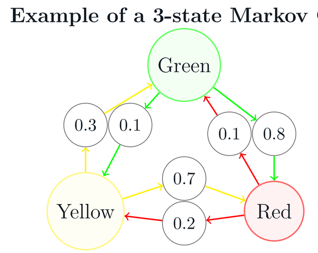
Introduction

In 2016, Pittsburgh Arlington PreK-8 implemented a disciplinary policy in which students are sorted into three categories: **green**, **yellow**, and **red**.

Information on the policy was not publicly available and led to the work [**1**] where the students movement through the colored categories was modeled as a **Markov Chain** with a random **transition function** (TF), leading to probabilistic modeling of the long-run proportion of students in each of the categories, where the long-run proportion of each category is itself a non-degenerate random variable.

The TFs studied in [**1**] were obtained by assuming that the entries of the TFs are truncated normal RVs from known distributions. Nine different models of this type were studied numerically. The typical result was that with probability of 90% or higher more than 10% of students ended up in the red category. Because of this result, the policy was deemed inadvisable in its current condition.

Example of a 3-state Markov Chain:



In this work we present an alternative, unified and mathematically tractable framework in which transition functions are sampled uniformly from the set of **reversible** transition functions.

Markov Chains

A (discrete-time, time-homogeneous, finite-state) **Markov Chain** (MC) is a sequence of random variables $X_0, X_1, \dots$ taking values in a finite set (state space), with the memoryless property: the probability that X_{n+1} is j depends on (=is a function of) the history $X_0, \dots, X_n$, only through the present,

$$\mathbb{P}(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = p(i,j).$$

The function p is the transition function, representing the probability of a "transition" from i to j.

A stationary distribution for the MC is a probability distribution π with the property that if X_0 is distributed according to π, then so is X_n, $\forall n \in \mathbb{N}$. This is equivalent to

$$\sum_i \pi(i)p(i,j) = \pi(j), \text{ for all } j. \tag{1}$$

The **Ergodic Theorem** for MCs states that under *mild conditions* (irreducibility and aperiodicity) on p

- the MC has a *unique* stationary distribution π; and
- the proportion of time the MC is in each state i tends to $\pi(i)$.

In what follows,

- *The transition function p is random*, and therefore π is a random variable.
- We are interested in the distribution of π.

Random Transition Functions

By saying that the TF is random we assume that it is sampled from some known distribution.

A MC is **reversible** if it essentially looks the same when moving forward and backward in time. The mathematical formulation is the following.

There exists a probability distribution π such that

$$\pi(i)p(i,j) = \pi(j)p(j,i), \quad \text{for all } i, j. \tag{2}$$

Note.

- (2) implies (1), therefore π is the stationary distribution.
- Not all TF satisfy (2).
- Every symmetric TF is reversible with π uniform.

Reversibility is a generalization of symmetry (it is essentially symmetry with respect to a weighted inner product).

Reversibility is a common occurrence in physical phenomena involving conservation laws. We also note that

- Reversibility yields a simpler expression for the stationary distribution, yet is not too restrictive or trivial.
- Shifting to the reversible framework allows puts the focus on structure of the TFs rather than on individual entries.

Our Model

Recall we are working with chains with three states. For simplicity, we'll assume the states are labeled 1, 2 and 3.

In the case of 3-state chains (2) gives

$$\begin{aligned}
\pi(1) &= \frac{p(2,1)p(3,1)}{p(3,1)p(1,2) + p(2,1)p(1,3) + p(2,1)p(3,1)}, \\
\pi(2) &= \frac{p(1,2)}{p(2,1)}\pi(1), \\
\pi(3) &= \frac{p(3,1)}{p(1,3)}\pi(1).
\end{aligned}$$

In our work we propose two alternatives to sampling from the set of reversible TFs:

- Sample from the set of symmetric irreducible matrices and transform to reversible through a change-of-basis transformation.
  → General, but requires solving eigenvalue equations.
- Sample uniformly from the ensemble of TFs and condition to be reversible.
  → Easier to apply for 3-state, described here.

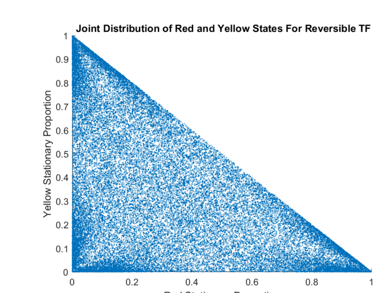
Sampling a Reversible TF Uniformly

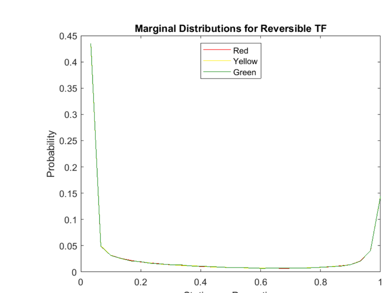
1. Sample first row uniformly from the simplex $\{(x,y,z) \in [0,1)^3 : x+y+z=1\}$.
2. Repeat for the second row.
3. Sample $p(3,1)$ uniformly from $[0, \frac{p(2,3)p(1,2)}{p(2,3)p(1,2)} + p(1,3)p(2,1)]$.
4. Set
$$\begin{aligned}
p(3,2) &= \frac{p(1,2)p(2,3)p(3,1)}{p(2,1)p(1,3)}, \text{ and} \\
p(3,3) &= 1 - p(3,1) - p(3,2)
\end{aligned}$$
5. Sample a permutation σ on $(1,2,3)$ uniformly, and change basis to $(e_{\sigma_1}, e_{\sigma_2}, e_{\sigma_3})$.

Note: all samples above are independent of each other.

Numerical Simulations

Below are the results of sampling 10^5 matrices using the above algorithm.





On the left is a scatter plot of the empirical joint distribution of $\pi(2)$ and $\pi(3)$. The distribution is not uniform, and more dense around near the corners.

On the right is the empirical density for $\pi(1), \pi(2), \pi(3)$ (all identical). The distribution is not uniform and is most dense near the extreme values 0 and 1, with highest density near zero.

In the context of the leveling policy, the empirical probability that in the long term at least 10% of the students are in the red state is around 49%, much lower than previous results, due high density near zero.

Final comments

- Analysis of π as the number of states tends to infinity is in the realm of random matrix theory, and is a direction for future research.
- From the modeling perspective, one may want to incorporate long-term memory or reinforcement. Research in this direction can be based on models exhibiting these features like Polya's Urn scheme.

References

[1] Murphy. T, Winger. A. (2018, June). *School Policy Evaluated with Markov Chain.*